Reflectionless measures

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Slide 1

What are we trying to accomplsh?

We would like to understand the geometric conditions that are imposed upon a non-atomic measure μ from the $L^2(\mu)$ boundedness of an associated Calderón-Zygmund operator.

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Notation

Fix $s \in (0, d)$. A Calderón-Zygmund kernel of dimension s is an odd function $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}^d$ satisfying

$$|K(x)| \leq \frac{1}{|x|^s} \text{ and } |\nabla K(x)| \leq \frac{1}{|x|^{s+1}}$$

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We say that a CZO (with CZ kernel K) is bounded in $L^2(\mu)$ if

$$\sup_{\varepsilon>0} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d \backslash B(x,\varepsilon)} K(x-y) f(y) d\mu(y) \Big|^2 d\mu(x) \leq C \|f\|^2_{L^2(\mu)}$$

for every $f \in L^2(\mu)$.

What would we like to know about μ?

– If $s \in \mathbb{Z}$, then we would like to determine whether μ is supported in some collection of Lipschitz surfaces (assuming that supp(μ) has dimension s). (Jones '89, David-Semmes '91, '93, Mattila-Melnikov-Verdera '96, David-Mattila '98, David-Leger '99, Nazarov-Tolsa-Volberg '12, Hofmann-Martel-Mayboroda-Uriate-Tuero '12).

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– If the CZO has non-integer dimension, then we would like to know sharp conditions on the density function of the measure. (Mateu-Prat-Verdera '05, Tolsa '11, Eiderman-Nazarov-Volberg '11, Reguera-Tolsa '14.)

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Theorem (Jaye-Nazarov-Reguera-Tolsa, '16)

Fix $s \in (d-1, d)$. Suppose that the s-Riesz transform (the CZO with kernel $K(x) = \frac{x}{|x|^{s+1}}$) is bounded in $L^2(\mu)$, then there is a constant $C > 0$ such that

$$\int_Q \int_0^\infty \Big(\frac{\mu(B(x,r)\cap Q)}{r^s}\Big)^2 \frac{dr}{r} d\mu(x) \leq C\mu(Q)$$

for every cube $Q \subset \mathbb{R}^d$.

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What is a reflectionless measure (associated to a integral kernel K)?

– It is a measure for which the potential

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is constant for x on the support of μ (when considered in a suitable weak sense).

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– Several general results (for both integer and non-integer homogeneity CZOs) in this spirit appear in *Reflectionless Measures for Calderón-Zygmund Operators II: Wolff potentials and rectifiability.* (J-Nazarov, arXiv September 2015.)
– We shall present a result in the opposite direction.

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A Case Study: Three revolutions (J-Nazarov arXiv:1307.3678)

Consider the kernel $K(z) = \frac{1}{|z|}\left(\frac{\bar{z}}{|z|}\right)^3$.

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$$\int_{B(z_0,r)} K(z-\omega)dm_2(\omega) = 0 \text{ for all } z \in B(z_0, r).$$

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Compare to **David-Leger:** If the Cauchy transform of a 1-dimensional non-atomic measure μ is bounded in $L^2(\mu)$, then the support of μ is rectifiable. This result was generalized by **Chousionis, Mateu, Prat, Tolsa** to other kernels.

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Construction of the measure

Take very fast decaying sequence $(r_n)_n$. First put $1/r_1$ roughly equally spaces discs $D_k^{(1)}$ of radius r_1 in $B(0,1)$. Then put r_1/r_2 roughly equally spaced discs $D_k^{(2)}$ of radius r_2 in each of the discs of radius r_1. Continue in this manner......

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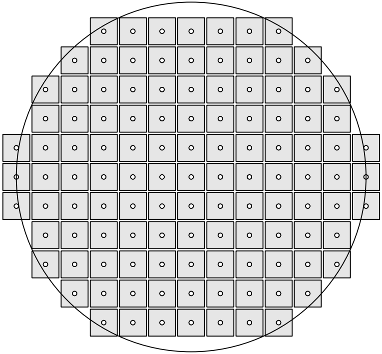

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We show that, for every generation n, and $z \in \text{supp}(\mu)$

$$\Big|\int_{D^n(z)\setminus D^{(n+1)}(z)} K(z-\xi)d\mu(\xi)\Big| \leq \sqrt{\frac{r_{n+1}}{r_n}}.$$

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For our dust K, $\mathcal{H}^1(K \cap \Gamma) = 0$ for any rectifiable curve Γ (just density considerations).

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Remark 1. For the Cantor dust measure μ, the limit

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Remarks!

Open Problem 1. Does there exist an AD-regular measure μ (this should satisfy, for some constant $C > 0$, $\frac{1}{C} r \leq \mu(B(x,r)) \leq Cr$ for all $x \in \text{supp}(\mu)$ and small $r > 0$) supported on an unrectifiable set K for which the three revolutions singular integral operator is bounded in $L^2(\mu)$?

The End