Harmonic Measure and 2-phase problems

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Harmonic Measure in higher dimensions

Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain. If the boundary of Ω is "regular," then we may always solve the **Dirichlet problem:** Given $f \in C(\partial\Omega)$, we may find $u_f \in C^2(\Omega) \cap C(\overline{\Omega})$ so that

1. $\triangle u_f = 0$ in Ω (i.e. u_f is **harmonic**)
2. $u_f(x) = f(x)$ for $x \in \partial\Omega$.

For $x \in \Omega$, the map $f \mapsto u_f(x)$ is a linear functional and defines a measure ω_Ω^x by the Riesz representation theorem.

$$u_f(x) = \int_{\partial\Omega} f d\omega_\Omega^x.$$

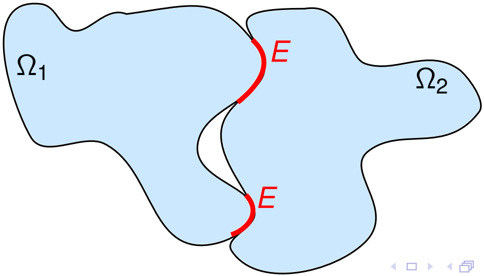
This agrees with the previous definition when we are in the complex plane.

General Question

Suppose Ω^1 and $\Omega^2 = (\overline{\Omega^1})^c$ are domains with harmonic measures ω^1 and ω^2. What geometric information can we get from knowing the relationship between ω^1 and ω^2?

Theorem (Bishop, Carleson, Garnett, Jones, '88)

Suppose $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ are simply connected Jordan domains and ω^i their harmonic measures. Let $E \subseteq \partial\Omega_1 \cap \partial\Omega_2$. Then $\omega^1|_E \perp \omega^2|_E$ if and only if E contains no 1-rectifiable set.



Tangent Measures

$$T_{\xi,r}(x) = \frac{x-\xi}{r}, \quad \text{so} \quad T_{\xi,r}(B(\xi,r)) = B(0,1).$$

Let μ be a measure in $\mathbb{R}^n$ and $\xi \in \operatorname{supp}\mu$.

$$T_{\xi,r}[\mu](A) = \mu(rA+\xi)$$

$$\operatorname{Tan}(\mu,\xi) = \left\{ \lim_{i\to\infty} c_i T_{\xi,r_i}[\mu] : c_i > 0, \ \ r_i \downarrow 0 \right\}.$$

Example: If $\mu = \mathscr{H}^d|_M$, M a smooth d-manifold and $\xi \in M$, then $\operatorname{Tan}(\mu,\xi) = \{c\mathscr{H}^d|_V : c > 0\}$ where V is the tangent plane to M at ξ.

Theorem (O'Neil, '95)

There is a Radon measure μ so that, for μ a.e. $\xi \in \mathbb{R}^n$, $\operatorname{Tan}(\mu,\xi)$ is all nonzero Radon measures.

Kenig-Toro

Theorem (Kenig, Preiss, Toro, '09)

Let Ω_1 and $\Omega_2 = (\Omega_1)^c$ be two NTA domains in $\mathbb{R}^{n+1}$, $n \geq 2$, and $\omega^i = \omega_{\Omega_i}^{x_i}$ their harmonic measures with poles at $x_i \in \Omega_i$. If $\omega^1 \ll \omega^2 \ll \omega^1$, then for a.e. $x \in \partial\Omega_1$,

$$\mathrm{Tan}(\omega^1, \xi) \subseteq \{c\mathscr{H}^n|_V : c > 0, V \text{ a } d\text{-plane}\}.$$

Theorem (Kenig, Toro, '06; Badger, '11; Badger, Engelstein, Toro, '15)

Let $\Omega_1 \subseteq \mathbb{R}^{n+1}$ and $\Omega_2 = (\overline{\Omega_1})^c$ be NTA. If $\log \frac{d\omega^2}{d\omega^1} \in VMO(d\omega^1)$, then there is $1 \leq d < n+1$ s.t. the boundary $\partial\Omega^1 = \Gamma_1 \cup \cdots \cup \Gamma_d$ where

$$\Gamma_k = \{\xi \in \partial\Omega^1 : \mathrm{Tan}(\omega^1, \xi) \subseteq \mathscr{F}_k\}.$$

More is true than discussed here, e.g. Γ_1 is open in $\partial\Omega^1$.

Idea of proof for Kenig-Toro

For a domain Ω and $x, y \in \Omega$, define the **Green function**

$$G_\Omega(x,y) = c_n|x-y|^{1-n} - \int_{\partial\Omega} c_d|x-\xi|^{1-n} d\omega_\Omega^y(\xi)$$

and $G_\Omega(x,y) = 0$ if x or $y \notin \Omega$. Then for $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\partial\Omega} \phi d\omega_\Omega^x = \int_\Omega \triangle\phi G_\Omega(x,y) dy.$$

Let $\Omega^i \subseteq \mathbb{R}^{n+1}$ be NTA. Let $\xi \in \partial\Omega^1$, $r_j \downarrow 0$, $\Omega_j^1 = T_{\xi,r_j}(\Omega^1)$.

Pass to a subsequence s.t. $\Omega_j^1 \to \Omega_\infty^1$,

$$\omega(B(\xi, r_j))^{-1} T_{\xi,r_j}[\omega^1] \to \omega_\infty^1 \in \mathrm{Tan}(\omega^1, \xi)$$

and there is u_∞^1 (zero off of Ω_∞^1) so that

$$\int \phi d\omega_\infty^1 = \int_{\Omega_\infty^1} \triangle\phi u_\infty^1(y) dy$$

Idea of Proof for Kenig-Toro

We can pass to a subsequence such that the same holds for ω^2. $\log \frac{d\omega^2}{d\omega^1} \in VMO$ implies $\omega^1_\infty = \omega^2_\infty$.
For $\phi \in C_c^\infty(\mathbb{R}^n)$, if $u_\infty = u^1_\infty - u^2_\infty$,

$$\int_{\mathbb{R}^n} \triangle\phi u_\infty = \int_{\Omega^1} \triangle\phi u^1_\infty - \int_{\Omega^2} \triangle\phi u^2_\infty = \int \phi d\omega^1_\infty - \int \phi d\omega^2_\infty = 0$$

Thus, u_∞ is harmonic, and $\omega^1_\infty = \omega_{u_\infty} = \frac{du_\infty}{d\nu}|_{u_\infty=0}$.

A bit more work shows that $\mathrm{Tan}(\omega^1, \xi) \subseteq \mathscr{P}_k$ with k depending on n and the NTA constants.

Theorem (Badger, '11)

If $\Omega \subseteq \mathbb{R}^{n+1}$ is NTA, $\xi \in \partial\Omega$, and $\mathrm{Tan}(\omega, \xi) \subseteq \mathscr{P}_k$, then $\mathrm{Tan}(\omega, \xi) \subseteq \mathscr{F}_\ell$ for some $\ell \leq k$.

Theorem (A., Mourgoglou, '16)

Same holds, but ω is **any** *Radon measure.*

Main Result

Definition

A domain $\Omega \subsetneq \mathbb{R}^{n+1}$ is Δ*-regular* if there is $R > 0$ so that

$$\sup_{\xi \in \partial\Omega} \sup_{x \in \partial B(\xi, r/2) \cap \Omega} \omega^x_{B \cap \Omega}(B(\xi, r) \cap \partial\Omega) \geq \beta > 0 \text{ for } r \in (0, R).$$

We say Ω is *2-sided* Δ*-regular* if $\mathrm{ext}(\Omega) := (\overline{\Omega})^c$ is also a connected Δ-regular domain.

Theorem (A., Mourgoglou, Tolsa, '16)

Let $\Omega^1 = (\Omega^2)_{ext} \subseteq \mathbb{R}^{n+1}$ be 2-sided Δ-regular domains. Let $\omega^i = \omega^{x_i}_{\Omega^i}$, $x_i \in \Omega_i$. Then $\omega^1 \perp \omega^2$ if and only if $\partial\Omega_1$ contains no n-rectifiable subset of positive n-measure.

Main Blackbox

Theorem (Girela-Sarrión, Tolsa, '16)

Let μ be Radon in $\mathbb{R}^{n+1}$, $B \subset \mathbb{R}^{n+1}$, s.t. $\mu(B) = r(B)^n$, and

(a) $\int_{B^c} |x - y|^{-n-1} d\mu \leq C_0$ *and* $\mu(B(x,r)) \leq C_0\, r^n$ *for all* $x \in B$ *and* $0 < r \leq r(B)$.

(b) $\exists$ *an n-plane L passing through the center of B s.t.* $\beta^L_{\mu,1}(B) \leq \delta$.

(c) $\mathcal{R}_{\mu|_B}$ *is bounded in* $L^2(\mu|_B)$ *with* $\|\mathcal{R}_{\mu|_B}\|_{L^2(\mu|_B)\to L^2(\mu|_B)} \leq C_1$.

(d) *For some constant* $0 < \tau \ll 1$,

$$\fint_B \left| \mathcal{R}\mu(x) - \fint_B (\mathcal{R}\mu)d\mu \right|^2 d\mu(x) \leq \tau.$$

If $0 < \delta, \tau \ll 1$ *(depending on* C_0 *and* C_1*), there is an n-rectifiable set* $\Gamma \subset \mathbb{R}^{n+1}$ *s.t.*

$$\mu(B \cap \Gamma) \gtrsim_{\delta,\tau} \mu(B).$$

Idea of Proof

Assume $\omega^1 \ll \omega^2 \ll \omega^1$.

- The ideas of Kenig and Toro extend to 2-sided Δ-regular domains (though not as nicely).
- In particular, if $\omega^1 \ll \omega^2 \ll \omega^1$, then ω^1-a.e., $\mathrm{Tan}(\omega^1, \xi)$ consists of flat measures, so in particular, $\beta_{\mu,1}(B(\xi, r)) \to 0$.
- For almost every ξ, we can find a shrinking sequence of balls centered at ξ where we can apply the theorem of Girela-Sarrión and Tolsa with $\mu = \omega^1$.

Theorem

Let Ω^1 *be a 2-sided* Δ*-regular domain in* $\mathbb{R}^{n+1}$ *and let* $\Omega^2 := ext(\Omega^1)$ *be its exterior. Let* ω^i *denote the harmonic measure of* Ω^i *with pole at some* $x^i \in \Omega^i$*. Assume* ω^i *are mutually absolutely continuous and set* $f = \frac{d\omega^2}{d\omega^1}$*. Let* $\xi \in \operatorname{supp} \omega^1$ *and assume*

$$\lim_{r\to 0} \left(\fint_{B(\xi,r)} f d\omega^1 \right) \exp\left(-\fint_{B(\xi,r)} \log f d\omega^1 \right) = 1.$$

If $\mathrm{Tan}(\omega^1,\xi) \neq \emptyset$*, then* $\mathrm{Tan}(\omega^1,\xi) \subseteq \mathscr{F}_k$ *for some* k *and*

$$\limsup_{r\to 0} \frac{\omega^1(B(\xi,2r))}{\omega^1(B(\xi,r))} < \infty,$$

Thank you!