Moduli of (complex) abelian varieties: homology and compactifications

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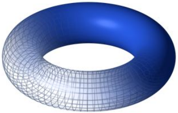
Raum, Zeit, Materie SFB seminar
January 5, 2016

(Complex) Elliptic curves = Riemann surfaces of genus one

- Geometrically: 

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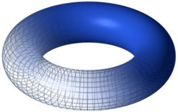
- Geometrically: 
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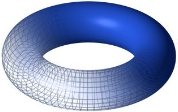
- Geometrically: 
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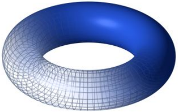
- Geometrically: 
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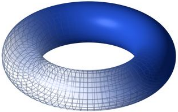
- Geometrically: 
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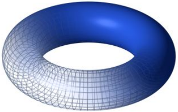
- Geometrically: 
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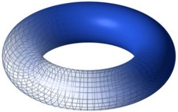
- Geometrically: 
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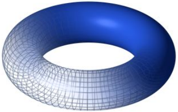
- Geometrically: 
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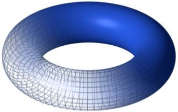
- Geometrically: 
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These are all equivalent!

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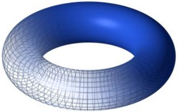
- Geometrically: 
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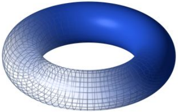
- Geometrically: 
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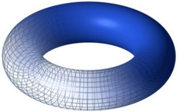
- Geometrically: 
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$$\tau' = (a\tau + b)(c\tau + d)^{-1}.$$

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Thus mark a point on E and require the automorphisms to fix it.

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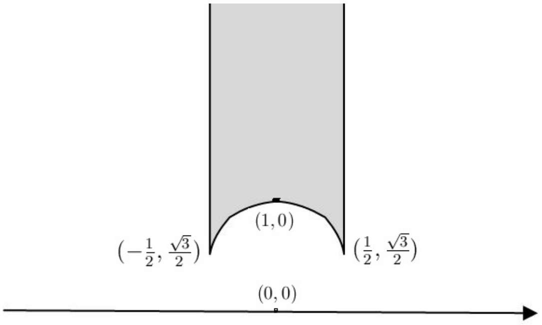

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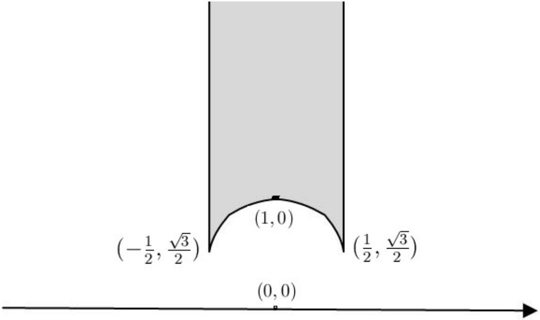

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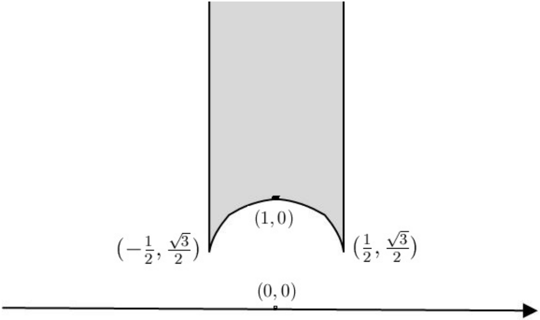

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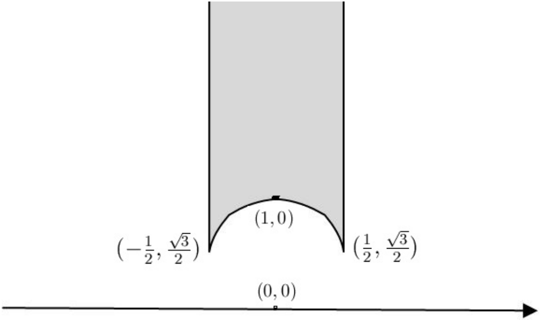

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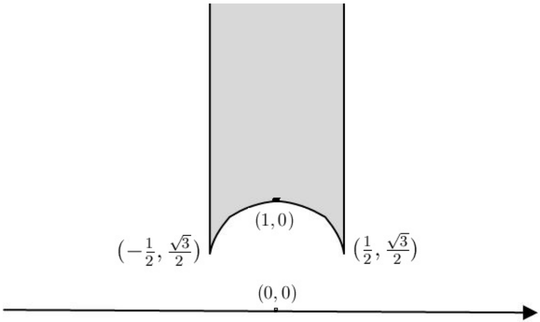

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- Compactified by adding the point at infinity, then

$$\mathcal{M}_{1,1} = \mathcal{A}_1 = \mathbb{P}^1$$

with three "special" points on $\mathbb{P}^1$.

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- Strong Faber's conjectures on the tautological ring.

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$\mathcal{A}_g$: the moduli space of principally polarized abelian varieties up to an algebraic isomorphism.

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Isomorphism of principally polarized abelian varieties: a biholomorphism that preserves polarization.

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- $\mathcal{A}_g$ is not compact.
- There are many approaches to compactifying $\mathcal{A}_g$!

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Theorem (Borel)

$H^k(\mathcal{A}_g, \mathbb{Q})$ is independent of g, for $g > k$, and is freely generated by $\{\lambda_{2i+1}\}$.

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Question

Why don't the λ_{2i} appear?

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Corollary

All even λ's can be expressed as polynomials in odd λ's:

$$\lambda_2 = \frac{\lambda_1^2}{2}, \qquad \lambda_4 = \lambda_1 \lambda_3 - \frac{\lambda_1^4}{8}, \qquad \ldots$$

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Proofs are topological: $\mathcal{M}_g = \mathcal{T}_g / MCG_g$, the Teichmüller space is contractible. Harer, Madsen-Weiss deal with $H^*(MCG_g)$.

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How to test the conjecture? Want to use intersection theory, but cannot on the open space $\mathcal{M}_g$. Intersection used for $\overline{\mathcal{M}}_g$.

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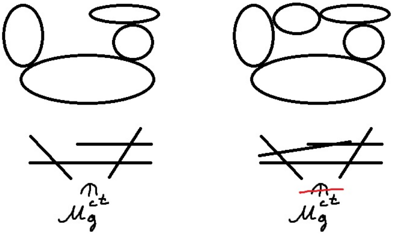

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The only relation in $R^*(\mathcal{A}_g^{\mathrm{Sat}})$ is the basic identity

$$(1 + \lambda_1 + \ldots + \lambda_g) \cdot (1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1.$$

Curiosity

Note that $R^*(\mathcal{A}_g^{\mathrm{Sat}}) = R^*(\mathcal{A}_{g+1})$. Why?

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Data for compactification: for each $k \le g$ a decomposition of $Sym^2_{\ge 0}(\mathbb{R}^k)$ into polyhedral cones, invariant under $GL_k(\mathbb{Z})$.

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- There exists a universal family of semiabelic varieties over $\mathcal{A}_g^{\mathrm{Vor}}$. [Alexeev]

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Note $\frac{k(k+1)}{2}$ are dimensions of boundary strata of $\mathcal{A}_g^{\text{Sat}}$...

Stable cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$

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Theorem (Charney-Lee)

The cohomology $H^k(\mathcal{A}_g^{\mathsf{Sat}})$ is independent of g for $g > k$, and is freely generated by $\lambda_1, \lambda_3, \lambda_5, \ldots$ and $\alpha_3, \alpha_5, \ldots$.

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Thus it is natural to still define the (algebraic) tautological ring of $\mathcal{A}_g^{\mathrm{Sat}}$ to be generated by λ_i.

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The class of the locus of products in $\mathcal{A}_4^{\text{Perf}}$ is tautological.
The (more or less) class of the locus of intermediate Jacobians of cubic threefolds is tautological in $\mathcal{A}_5^{\text{Perf}}$.

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Question

Is there any reasonable compactification of $\mathcal{M}_g$ whose homology stabilizes?

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Trying to explain the phenomena

$\mathcal{A}_g^{\mathrm{Sat}}, \mathcal{A}_g^{\mathrm{Perf}}, \mathcal{A}_g^{\mathrm{Vor}}$ are singular (even as stacks/orbifolds).

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- For X compact, $IH^*(X)$ satisfies Poincaré duality.
- For any X, have $IH_k(X) \to H_k(X)$, such that the image is contained in the set of algebraic classes.

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Is it equal to stable $H^k(\mathcal{A}_g^{\mathrm{Perf}})$ or to stable $H^{g(g+1)-k}(\mathcal{A}_g^{\mathrm{Perf}})$?