Moduli of (complex) abelian varieties: homology and compactifications

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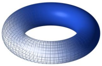
(Complex) Elliptic curves = Riemann surfaces of genus one

- Geometrically: 
- Algebraically: E_λ = closure of $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{C}^2$
- Analytically: $E = \mathbb{C}/\Lambda$, for Λ a lattice of full rank:
  $\Lambda \approx \mathbb{Z}^2$; $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$; So $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$.

When are two elliptic curves equal?biholomorphic?
What does "equal" mean?

As complex manifolds, biholomorphic?
...or isomorphic as algebraic varieties?
...or as lattices?

These are all equivalent!

Theorem

$E_\lambda \approx E_{\lambda'}$ if and only if $j(\lambda) = j(\lambda')$.

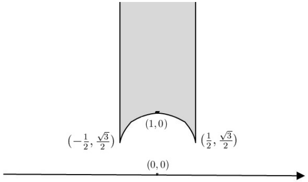
Theorem

*Any holomorphic map $E_\tau \to E_{\tau'}$ lifts to a linear map $\mathbb{C} \to \mathbb{C}$.
Then $E_\tau \approx E_{\tau'}$ if and only if $\exists \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ such that*
$\tau' = (a\tau + b)(c\tau + d)^{-1}$

Moduli of (complex) elliptic curves with a marked point

Difficulty: any elliptic curve has infinitely many automorphisms $z \mapsto z + a$, for any $a \in \mathbb{C}$.

Thus mark a point on E and require the automorphisms to fix it.



- Global geometry not immediately visible.
- Orbifold points $\tau = e^{2\pi i/3}$ and $\tau = i$: extra automorphisms.
- The moduli space is not compact.
- Compactified by adding the point at infinity, then

$$\mathcal{M}_{1,1} = \mathcal{A}_1 = \mathbb{P}^1$$

with three "special" points on $\mathbb{P}^1$.

Generalizing moduli of elliptic curves.
Approach 1: Riemann surfaces

$\mathcal{M}_g$:=moduli of compact Riemann surfaces of genus $g \geq 1$, up to biholomorphism.

- A Riemann surface of genus $g > 1$ has at most $84(g-1)$ automorphisms, thus no need to mark any points to get a good moduli space.
- $\mathcal{M}_g$ is a complex orbifold of dimension $3g-3$ [Riemann].
- $\mathcal{M}_g$ has a nice *Deligne-Mumford compactification* $\overline{\mathcal{M}}_g$, which is a smooth orbifold, with simple normal crossing boundary.
- Geometry and topology of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are studied extensively.
- The homology or Chow rings of $\mathcal{M}_g$ or $\overline{\mathcal{M}}_g$ are very difficult and very big, but there is a natural tautological subring.
- Strong Faber's conjectures on the tautological ring.

Generalizing moduli of elliptic curves. Approach 2: abelian varieties algebraically

Abelian variety: a projective g-dimensional variety A (a compact submanifold of $\mathbb{CP}^N$), group structure on points.

Principal polarization: the first Chern class of an ample line bundle Θ with one section.

(Ample means has positive curvature; equivalently, the space of sections of $\Theta^{\otimes n}$ embeds A into $\mathbb{CP}^N$, for n large enough)

(for $g = 1$, this is just one point on A)

$\mathcal{A}_g$: the moduli space of principally polarized abelian varieties up to an algebraic isomorphism.

Generalizing moduli of elliptic curves. Approach 3: complex abelian varieties analytically

Abelian variety $A_\tau := \mathbb{C}^g / \mathbb{Z}^g + \mathbb{Z}^g \tau$, where the
Period matrix τ lies in the
Siegel upper half-space

$$\mathcal{H}_g := \{\tau \in Mat_{g\times g}(\mathbb{C}) \,|\, \tau = \tau^t, Im\,\tau > 0\}$$

Polarization Θ_τ: the zero locus in A_τ of the theta function

$$\theta(z) := \sum_{n\in\mathbb{Z}^g} \exp\left((\pi i\, n^t(\tau n + 2z)\right).$$

Isomorphism of principally polarized abelian varieties: a biholomorphism that preserves polarization.

Moduli of abelian varieties, complex-analytically

Theorem

Any holomorphic map $A_\tau \to A_{\tau'}$ lifts to a linear holomorphic map $\mathbb{C}^g \to \mathbb{C}^g$. It follows that

$$\mathcal{A}_g = Sp(2g, \mathbb{Z}) \backslash \mathcal{H}_g$$

where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \circ \tau = (C\tau + D)^{-1}(A\tau + B)$.

Properties of $\mathcal{A}_g$:

- Smooth orbifold: for any τ, $Stab(\tau) \subset \mathrm{Sp}(2g, \mathbb{Z})$ is finite.
- $\dim_{\mathbb{C}} \mathcal{A}_g = \frac{g(g+1)}{2} = \dim_{\mathbb{C}}(\text{symmetric } Mat_{g\times g}(\mathbb{C}))$.
- $H^*(\mathcal{A}_g) = H^*(\mathrm{Sp}(2g, \mathbb{Z}))$ in general is extremely complicated.
- $\mathcal{A}_g$ is not compact.
- There are many approaches to compactifying $\mathcal{A}_g$!

Stable cohomology of $\mathcal{A}_g$

Hodge vector bundle: the rank g vector bundle $\mathbb{E} \to \mathcal{A}_g$ of holomorphic 1-forms: it has fiber $H^{1,0}(A)$ over $[A]$.

Hodge classes $\lambda_i := c_i(\mathbb{E}) \in H^{2i}(\mathcal{A}_g, \mathbb{Q})$ the Chern classes of the Hodge bundle (also in Chow $CH^i(\mathcal{A}_g)$).

Theorem (Borel)

$H^k(\mathcal{A}_g, \mathbb{Q})$ is independent of g, for $g > k$, and is freely generated by $\{\lambda_{2i+1}\}$.

Borel's proof is about group cohomology of $Sp(2g, \mathbb{Z})$.
Since $\mathcal{H}_g$ is contractible, $H^*(\mathcal{A}_g) = H^*(Sp(2g, \mathbb{Z}))$.

(Of course no approach in sight to stabilization of $CH^k(\mathcal{A}_g)$)

Question

Why don't the λ_{2i} appear?

Relation among the Hodge classes on $\mathcal{A}_g$

$\mathbb{E} \oplus \overline{\mathbb{E}}$ is the rank $2g$ bundle over $\mathcal{A}_g$, with fiber

$$H^1(A, \mathbb{C}) = H^{1,0}(A, \mathbb{C}) \oplus H^{0,1}(A, \mathbb{C}).$$

Thus $c_i(\mathbb{E} \oplus \overline{\mathbb{E}}) = 0$ for $i > 0$.

Theorem (Mumford's Basic identity)

$$(1 + \lambda_1 + \ldots + \lambda_g) \cdot (1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1 \in H^*(\mathcal{A}_g).$$

Corollary

All even λ's can be expressed as polynomials in odd λ's:

$$\lambda_2 = \frac{\lambda_1^2}{2}, \qquad \lambda_4 = \lambda_1 \lambda_3 - \frac{\lambda_1^4}{8}, \qquad \ldots$$

Stable cohomology of $\mathcal{M}_g$

Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ sends a Riemann surface to its Jacobian. Hodge bundle and classes pull back, the basic identity pulls back.

Theorem (Harer)

$H^k(\mathcal{M}_g, \mathbb{Q})$ is independent of g, for $g \gg k$.

Theorem (Madsen-Weiss [Mumford's conjecture])

$H^k(\mathcal{M}_g)$ is freely generated by $\kappa_i \in H^{2i}(\mathcal{M}_g)$ for $g > 3k$.

Mumford-Morita-Miller kappa classes:
$\Psi :=$(c_1 of) the line bundle over $\mathcal{M}_{g,1}$ with $\Psi|_{X,p} = T_p^* X$.
$\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ the forgetful map; $\kappa_i := \pi_*(\Psi^{i+1})$.

Proofs are topological: $\mathcal{M}_g = \mathcal{T}_g / MCG_g$, the Teichmüller space is contractible. Harer, Madsen-Weiss deal with $H^*(MCG_g)$.

Tautological rings of $\mathcal{A}_g$ and $\mathcal{M}_g$

λ_i on $\mathcal{A}_g$ and κ_i on $\mathcal{M}_g$ are defined also outside of stable range.

Tautological ring $R^*(\mathcal{A}_g)$: subring of cohomology generated by λ_i.
Tautological ring $R^*(\mathcal{M}_g)$: subring of cohomology generated by κ_i.
(Should also consider these as subrings in the Chow).

Theorem (van der Geer)

The only relations in $R^*(\mathcal{A}_g)$ are $\lambda_g = 0$ and the basic identity

$$(1+\lambda_1+\ldots+\lambda_g)\cdot(1-\lambda_1+\ldots+(-1)^g\lambda_g)=1.$$

$\Longrightarrow R^*(\mathcal{A}_g)$ has Poincaré duality with socle in dimension $2\cdot\frac{g(g-1)}{2}$.

Faber's conjecture

$R^*(\mathcal{M}_g)$ has Poincaré duality with socle in dimension $2\cdot(g-2)$.

Faber's conjecture: status and corollaries

Faber's conjecture

$R^*(\mathcal{M}_g)$ has Poincaré duality with socle in complex dimension $2\cdot(g-2)$ $g-2$.

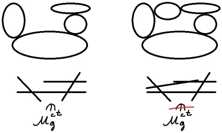
- Vanishing: $R^k(\mathcal{M}_g)=0$ for $k>g-2$.
  True [Ionel, Looijenga, Graber-Vakil, ...]
- Socle: $R^{g-2}(\mathcal{M}_g)=\mathbb{Q}$.
  True [Faber, Looijenga]
- Perfect Pairing: $R^k(\mathcal{M}_g)\times R^{g-2-k}(\mathcal{M}_g)\to R^{g-2}(\mathcal{M}_g)=\mathbb{Q}$ is a perfect pairing, $R^k=(R^{g-k})^*$.
  Not known! Fails for the analog for $\overline{\mathcal{M}}_{g,n}, \mathcal{M}^{ct}_{g,n}$
  [Petersen, Petersen-Tommasi]

Conjecture says $R^*(\mathcal{M}_g)$ "looks like" cohomology of a compact X of dimension $2\cdot(g-2)$, with no odd cohomology. What is X?
How to test the conjecture? Want to use intersection theory, but cannot on the open space $\mathcal{M}_g$. Intersection used for $\overline{\mathcal{M}}_g$.

Compactifying $\mathcal{M}_g$

Deligne-Mumford compactification $\overline{\mathcal{M}}_g$: boundary is a collection of irreducible divisors, normal crossing.

Curves of compact type: $\mathcal{M}_g^{ct} = \overline{\mathcal{M}}_g \setminus \delta_0$.



Tautological rings of $\overline{\mathcal{M}}_g$ and $\mathcal{M}_g^{ct}$: generated by κ_i, all boundary strata, κ_i and Ψ pushed from the boundary, . . .

Faber's questions

Does $R^*(\overline{\mathcal{M}}_g)$ have duality with socle in dimension $3g-3$?

Compactifying $\mathcal{A}_g$: Satake-Baily Borel compactification

Satake compactification: As a set, $\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_0$.

To put scheme structure: $\lim_{t\to\infty} \begin{pmatrix} it & z^t \\ z & \tau' \end{pmatrix} := \tau'$.

More generally, cross out all rows and columns with infinities (in fact, take out the kernel of $Im\,\tau$):

$$\lim_{t_1,t_2\to\infty} \begin{pmatrix} \tau_1 & * & * & * & \tau_2 \\ * & * & it_1 & * & * \\ * & it_1 & * & * & * \\ * & * & * & it_2 & * \\ \tau_2^t & * & * & * & \tau_3 \end{pmatrix} := \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2^t & \tau_3 \end{pmatrix}.$$

- As a set, $\mathcal{A}_g^{\mathrm{Sat}}$ is very easy to describe.
- There is no reasonable universal family of abelian varieties over $\mathcal{A}_g^{\mathrm{Sat}}$.
- $\mathcal{A}_g^{\mathrm{Sat}}$ is very singular, boundary is codimension g.

Tautological ring of $\mathcal{A}_g^{\mathsf{Sat}}$

$R^*(\mathcal{A}_g^{\mathsf{Sat}})$ is the ring generated by Hodge classes λ_i.

Theorem (Ekedahl-Oort)

The class of $\mathcal{A}_{g-1} \subset \mathcal{A}_g^{Sat}$ is a multiple of λ_g.

Theorem (van der Geer in H^*, Esnault-Viehweg in CH^*)

The only relation in $R^*(\mathcal{A}_g^{\mathsf{Sat}})$ is the basic identity

$$(1+\lambda_1+\ldots+\lambda_g)\cdot(1-\lambda_1+\ldots+(-1)^g\lambda_g)=1.$$

Curiosity

Note that $R^*(\mathcal{A}_g^{\mathsf{Sat}}) = R^*(\mathcal{A}_{g+1})$. Why?

Toroidal compactifications of $\mathcal{A}_g$

Idea: bigger than $\mathcal{A}_g^{\mathsf{Sat}}$, with a universal family.

Universal family of abelian varieties $\mathcal{X}_g \to \mathcal{A}_g$: fiber A over $[A]$.

Then set $\lim_{t\to\infty}\begin{pmatrix} it & z^t \\ z & \tau' \end{pmatrix} := (\tau', z) \in \mathcal{X}_{g-1}$.

So $\mathcal{A}_g^{Tor} = \mathcal{A}_g \sqcup \mathcal{X}_{g-1} \sqcup ???$. How to continue further? Maybe

$$\lim_{t_1,t_2\to\infty}\begin{pmatrix} it_1 & x & z_1^t \\ x & it_2 & z_2^t \\ z_1 & z_2 & \tau' \end{pmatrix} := (\tau', z_1, z_2) \in \mathcal{X}_{g-2}^{\times 2} \ ?$$

No good! Codimension 2 degeneration, need to record x.
Correct approach: don't go to infinity, consider $Ker(Im(\tau))$.

Data for compactification: for each $k \le g$ a decomposition of $Sym^2_{\ge 0}(\mathbb{R}^k)$ into polyhedral cones, invariant under $GL_k(\mathbb{Z})$.

Toroidal compactifications $\mathcal{A}_g^{\mathrm{Perf}}$ and $\mathcal{A}_g^{\mathrm{Vor}}$

Perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}$

- The boundary $\partial\mathcal{A}_g^{\mathrm{Perf}}$ is irreducible, $\mathcal{X}_{g-1}$ is dense within it.
- Maps to $\mathcal{A}_g^{\mathrm{Sat}}$, the structure over $\mathcal{A}_{g-k}$ is some toric variety bundle over $\mathcal{X}_{g-k}^{\times k}$ independent of g — only depends on k.
- Is the canonical model of $\mathcal{A}_g$ for $g \geq 12$ for the minimal model program, i.e. $K_{\mathcal{A}_g^{\mathrm{Perf}}}$ is ample. [Shepherd-Barron]
- No known universal family over $\mathcal{A}_g^{\mathrm{Perf}}$.

Second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$

- The boundary $\partial\mathcal{A}_g^{\mathrm{Vor}}$ has many (likely $\gg g$) irreducible divisorial components.
- Maps to $\mathcal{A}_g^{\mathrm{Sat}}$, exist boundary divisors mapping to $\mathcal{A}_k$ for small k.
- There exists a universal family of semiabelic varieties over $\mathcal{A}_g^{\mathrm{Vor}}$. [Alexeev]

Intersection theory of divisors on $\mathcal{A}_g^{Tor}$

$L := \lambda_1$; D :=the sum of all boundary divisors.
(L and D span $H^2(\mathcal{A}_g^{\mathrm{Perf}}) = \mathrm{Pic}(\mathcal{A}_g^{\mathrm{Perf}})$)

Conjecture [G.-Hulek]

The intersection number $L^a D^{\frac{g(g+1)}{2}-a}$ is zero unless $a = \frac{k(k+1)}{2}$.

Theorem (Erdenberger-G.-Hulek)

The conjecture holds for $g \leq 4$ for any a.

Theorem (G.-Hulek)

The conjecture holds for $a > \frac{(g-3)(g-2)}{2}$ for any g.

Any reason for this to hold?

Note $\frac{k(k+1)}{2}$ are dimensions of boundary strata of $\mathcal{A}_g^{\mathrm{Sat}}$...

Stable cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$

Theorem (Charney-Lee)

The cohomology $H^k(\mathcal{A}_g^{\mathrm{Sat}})$ is independent of g for $g > k$, and is freely generated by $\lambda_1, \lambda_3, \lambda_5, \ldots$ and $\alpha_3, \alpha_5, \ldots$.

Proof purely topological.

Theorem (Chen-Looijenga)

No polynomial in the classes α_i is algebraic.

Also gives a more algebraic proof.

Thus it is natural to still define the (algebraic) tautological ring of $\mathcal{A}_g^{\mathrm{Sat}}$ to be generated by λ_i.

Stable cohomology of $\mathcal{A}_g^{\mathrm{Perf}}$

Theorem (G.-Hulek-Tommasi)

The cohomology $H^{g(g+1)-k}(\mathcal{A}_g^{\mathrm{Perf}})$ is independent of g for $g > k$, and is purely algebraic.

$\mathcal{A}_g^{\mathrm{Perf}}$ is singular, so there is no Poincaré duality, can have

$$H^{g(g+1)-k}(\mathcal{A}_g^{\mathrm{Perf}}) \not\cong H_k(\mathcal{A}_g^{\mathrm{Perf}}).$$

Smooth matroidal locus $\mathcal{A}_g^{\mathrm{Matr}} = \mathcal{A}_g^{\mathrm{Perf}} \cap \mathcal{A}_g^{\mathrm{Vor}}$. [Melo-Viviani]

Theorem (G.-Hulek-Tommasi)

The cohomology $H^k(\mathcal{A}_g^{\mathrm{Matr}})$ is independent of g for $g > k$, and is purely algebraic.

Extended tautological ring

Dream

- Prove that $H^k(\mathcal{A}_g^{\mathrm{Perf}})$ stabilizes.
  [J. Giansiracusa-Sankaran, in progress]
- Understand the stable failure of Poincaré duality on $\mathcal{A}_g^{\mathrm{Perf}}$.
- Understand the algebraic generators x_i of stable cohomology.
- Define extended tautological ring of $\mathcal{A}_g^{\mathrm{Perf}}$, generated by x_i.
- Formulate an analog of extended Faber's conjecture.
- Prove that the extended tautological ring contains the classes of natural geometric subvarieties, starting with $\mathcal{A}_i^{Perf} \times \mathcal{A}_{g-i}^{Perf}$.

Theorem (G.-Hulek)

The class of the locus of products in $\mathcal{A}_4^{\mathrm{Perf}}$ is tautological.
The (more or less) class of the locus of intermediate Jacobians of cubic threefolds is tautological in $\mathcal{A}_5^{\mathrm{Perf}}$.

Stable cohomology of $\overline{\mathcal{M}}_g$ or $\mathcal{A}_g^{\mathrm{Vor}}$?

Since $\dim H^2(\overline{\mathcal{M}}_g) = 1 + \lfloor g/2 \rfloor$, can't have stabilization.

Conjecturally, $\dim H^2(\mathcal{A}_g^{\mathrm{Vor}}) \gtrsim g$, so no stabilization either.

Maybe other compactifications of $\mathcal{M}_g$?

The Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ extends to $\overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{Perf}}$.
[Alexeev-Brunyate].

However, $\mathcal{M}_g^{ct} \to \mathcal{A}_g$, contracts each δ_i to a codimension 3 locus.
Thus H^6 of the image does not stabilize.

Question

Is there any reasonable compactification of $\mathcal{M}_g$ whose homology stabilizes?

Trying to explain the phenomena

$\mathcal{A}_g^{\mathsf{Sat}}, \mathcal{A}_g^{\mathsf{Perf}}, \mathcal{A}_g^{\mathsf{Vor}}$ are singular (even as stacks/orbifolds).

Goresky-Macpherson intersection homology for singular spaces.

- For smooth X, $IH^*(X) = H^*(X)$; so $IH^*(\mathcal{A}_g) = H^*(\mathcal{A}_g)$.
- For X compact, $IH^*(X)$ satisfies Poincaré duality.
- For any X, have $IH_k(X) \to H_k(X)$, such that the image is contained in the set of algebraic classes.

Stable intersection homology

Theorem [Borel+Looijenga, Saper-Stern]

The stable intersection cohomology of $\mathcal{A}_g^{\mathsf{Sat}}$ is equal to the stable cohomology of $\mathcal{A}_g$ (i.e. is generated by λ_{2i+1}).

(Recall that stable $H^k(\mathcal{A}_g^{\mathsf{Sat}})$ is freely generated by $\lambda_1, \lambda_3, \lambda_5, \ldots$ and $\alpha_3, \alpha_5, \ldots$, and that no polynomial in α_i is algebraic.)

[Charney-Lee, Looijenga]

Theorem (G.-Hulek)

For $g \leq 4$, $IH^*(\mathcal{A}_g^{\mathsf{Sat}}) = R^*(\mathcal{A}_g^{\mathsf{Sat}})$, except possibly for $IH^{10}(\mathcal{A}_4^{\mathsf{Sat}})$.

Question

Is there a stable decomposition theorem for $\mathcal{A}_g^{\mathsf{Perf}} \to \mathcal{A}_g^{\mathsf{Sat}}$?
Does $IH^k(\mathcal{A}_g^{\mathsf{Perf}})$ stabilize?
Is it equal to stable $H^k(\mathcal{A}_g^{\mathsf{Perf}})$ or to stable $H^{g(g+1)-k}(\mathcal{A}_g^{\mathsf{Perf}})$?