Effective algebraic Schottky problem

Samuel Grushevsky
Princeton University

Plan:

1) What is Schottky problem / why is it hard and why is it useful?

2) Notations and definitions

3) History of the problem, and different approaches to solving it

Plan:

1) What is Schottky problem / why is it hard and why is it useful?

2) Notations and definitions

3) History of the problem, and different approaches to solving it

4) Effective algebraic solution

5) (*towards*) Explicit equations for theta functions

Schottky problem is the following question:

which principally polarized abelian varieties are Jacobians of curves?

It relates two important moduli/classification spaces, and would allow one to relate results in algebraic geometry to number theory and modular forms.

Schottky problem is the following question:

which principally polarized abelian varieties are Jacobians of curves?

It relates two important moduli/classification spaces, and would allow one to relate results in algebraic geometry to number theory and modular forms.

A good understanding of the answer could perhaps lead to a better geometric understanding of abelian varieties starting from curves, or allow one to relate the cohomology and other geometric properties of the two moduli spaces.

Notations:

$\mathcal{M}_g$ — moduli space of Riemann surfaces of genus g
$\mathcal{A}_g$ — moduli space of principally polarized abelian varieties
$\mathcal{H}_g$ — Siegel upper half-space for dimension g

$$J : \mathcal{M}_g \to \mathcal{A}_g \qquad \mathcal{J}_g := J(\mathcal{M}_g) \qquad \mathcal{A}_g = \mathcal{H}_g / \operatorname{Sp}(2g, \mathbf{Z})$$

Notations:

$\mathcal{M}_g$ — moduli space of Riemann surfaces of genus g
$\mathcal{A}_g$ — moduli space of principally polarized abelian varieties
$\mathcal{H}_g$ — Siegel upper half-space for dimension g

$$J : \mathcal{M}_g \to \mathcal{A}_g \qquad \mathcal{J}_g := J(\mathcal{M}_g) \qquad \mathcal{A}_g = \mathcal{H}_g / \mathrm{Sp}(2g, \mathbf{Z})$$

g	$\dim \mathcal{M}_g$		$\dim \mathcal{A}_g$	
1	1	$=$	1	
2	3	$=$	3	
3	6	$=$	6	
4	9	$+1 =$	10	Schottky's original equation
5	12	$<$	15	Partial geometric results
g	$3g-3$	$<<$	$\frac{g(g+1)}{2}$	???

Theta functions with characteristics

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z) := \sum_{n\in\mathbf{Z}^g} \exp\left[\Big(n+\varepsilon/2, \tau(n+\varepsilon/2)\Big) + 2\Big(n+\varepsilon/2, z+\delta/2\Big)\right]$$

and theta functions of the second order

$$\Theta[\varepsilon](\tau,z) := \theta\begin{bmatrix}\varepsilon\\ 0\end{bmatrix}(2\tau,2z)$$

Theta functions with characteristics

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z) := \sum_{n\in\mathbf{Z}^g} \exp\left[\Big(n+\varepsilon/2, \tau(n+\varepsilon/2)\Big) + 2\Big(n+\varepsilon/2, z+\delta/2\Big)\right]$$

and theta functions of the second order

$$\Theta[\varepsilon](\tau,z) := \theta\begin{bmatrix}\varepsilon\\ 0\end{bmatrix}(2\tau,2z)$$

are related via Riemann's bilinear addition theorem

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(z)\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(w) = \sum_{\sigma\in(\mathbf{Z}/2\mathbf{Z})^g} (-1)^{(\delta,\sigma)}\Theta[\sigma+\varepsilon]\left(\frac{z+w}{2}\right)\Theta[\sigma]\left(\frac{z-w}{2}\right)$$

The level subgroups of the modular group are

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Sp}(2g, \mathbf{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \,\middle|\, \mathsf{diag}(a^t b) \equiv \mathsf{diag}(c^t d) \equiv 0 \bmod 2n \right\}.$$

The level subgroups of the modular group are

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Sp}(2g, \mathbf{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \,\middle|\, \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \mod 2n \right\}.$$

These are normal in $\mathsf{Sp}(2g, \mathbf{Z})$ and their index was computed by *Igusa*. In particular

$$N_g := \#\left(\mathsf{Sp}(2g, \mathbf{Z})/\Gamma_g(2,4)\right) = 2^{g^2+2g} \prod_{k=1}^{g} (2^{2k} - 1) \sim 2^{2g^2}$$

We denote $\mathcal{A}_g^{2,4} := \mathcal{H}_g/\Gamma_g(2,4)$ and similarly define $\mathcal{M}_g^{2,4}$.

As functions of z for a fixed τ, the functions $\Theta[\varepsilon](\tau, z)$ define the Kummer map of $A_\tau := \mathbf{C}^g/\mathbf{Z}^g + \tau\mathbf{Z}^g$, i.e.

$$\begin{array}{rccc} K: & \mathcal{A}_\tau/\pm 1 & \to & \mathbf{P}^{2^g-1} \\ & z & \to & \left\{\Theta[\varepsilon](\tau, z)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

As functions of z for a fixed τ, the functions $\Theta[\varepsilon](\tau,z)$ define the Kummer map of $A_\tau := \mathbf{C}^g/\mathbf{Z}^g + \tau\mathbf{Z}^g$, i.e.

$$\begin{array}{rccc} K: & \mathcal{A}_\tau/\pm 1 & \to & \mathbf{P}^{2^g-1} \\ & z & \to & \left\{\Theta[\varepsilon](\tau,z)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

Theta constants $\Theta[\varepsilon](\tau,0)$ are modular with respect to $\Gamma(2,4)$ of weight one half and thus define the map

$$\begin{array}{rccc} Th: & \mathcal{A}_g^{2,4} & \to & \mathbf{P}^{2^g-1} \\ & \tau & \to & \left\{\Theta[\varepsilon](\tau,0)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

which is known (*R. Salvati Manni*) to be generically injective, and is always at most finite-to-one.

As functions of z for a fixed τ, the functions $\Theta[\varepsilon](\tau, z)$ define the Kummer map of $A_\tau := \mathbf{C}^g/\mathbf{Z}^g + \tau\mathbf{Z}^g$, i.e.

$$\begin{array}{rccc} K: & \mathcal{A}_\tau/\pm 1 & \to & \mathbf{P}^{2^g-1} \\ & z & \to & \left\{\Theta[\varepsilon](\tau, z)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

Theta constants $\Theta[\varepsilon](\tau, 0)$ are modular with respect to $\Gamma(2, 4)$ of weight one half and thus define the map

$$\begin{array}{rccc} Th: & \mathcal{A}_g^{2,4} & \to & \mathbf{P}^{2^g-1} \\ & \tau & \to & \left\{\Theta[\varepsilon](\tau, 0)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

which is known (*R. Salvati Manni*) to be generically injective, and is always at most finite-to-one.

Algebraic Schottky problem:

describe $Th(J(\mathcal{M}_g^{2,4})) \subset Th(\mathcal{A}_g^{2,4})$

History of the problem

1880s: *Schottky* (+1900s *Jung* + perhaps *Riemann* earlier)
Take $\tilde{C} \to C$ an unramified double cover. Let τ be the period matrix of C and let π be the period matrix of the Prym. Then

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}^2 (\pi, 0) = \theta \begin{bmatrix} 0\,\varepsilon \\ 0\,\delta \end{bmatrix} (\tau, 0) \theta \begin{bmatrix} 0\,\varepsilon \\ 1\,\delta \end{bmatrix} (\tau, 0)$$

History of the problem

1880s: *Schottky* (+1900s *Jung* + perhaps *Riemann* earlier)
Take $\tilde{C} \to C$ an unramified double cover. Let τ be the period matrix of C and let π be the period matrix of the Prym. Then

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}^2 (\pi, 0) = \theta \begin{bmatrix} 0 \, \varepsilon \\ 0 \, \delta \end{bmatrix} (\tau, 0) \theta \begin{bmatrix} 0 \, \varepsilon \\ 1 \, \delta \end{bmatrix} (\tau, 0)$$

Using this (together with Riemann's addition theorem) allows us to get some equations for $Th(\mathcal{J}_g^{2,4})$ from equations for $Th(\mathcal{A}_{g-1}^{2,4})$.

History of the problem

1880s: *Schottky* (+1900s *Jung* + perhaps *Riemann* earlier)
Take $\tilde{C} \to C$ an unramified double cover. Let τ be the period matrix of C and let π be the period matrix of the Prym. Then

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}^2 (\pi, 0) = \theta \begin{bmatrix} 0 \, \varepsilon \\ 0 \, \delta \end{bmatrix} (\tau, 0) \theta \begin{bmatrix} 0 \, \varepsilon \\ 1 \, \delta \end{bmatrix} (\tau, 0)$$

Using this (together with Riemann's addition theorem) allows us to get some equations for $Th(\mathcal{J}_g^{2,4})$ from equations for $Th(\mathcal{A}_{g-1}^{2,4})$.

1960s: *H. Farkas and Rauch*
prove the validity of this approach, and show that some non-trivial equations result.

1980s: **Theorem** (*van Geemen / Donagi*).
If we do this for all / for just one double cover and write down all the resulting Schottky-Jung relations (using the full ideal of $Th(\mathcal{A}_{g-1}^{2,4})$), Jacobians will be an irreducible component of the solution set.

1980s: **Theorem** (*van Geemen / Donagi*).
If we do this for all / for just one double cover and write down all the resulting Schottky-Jung relations (using the full ideal of $Th(\mathcal{A}_{g-1}^{2,4})$), Jacobians will be an irreducible component of the solution set.

+: We get explicit algebraic equations for theta constants.
In fact (*Schottky, Farkas, Rauch, Igusa, Mumford*) we do get the one defining equation in genus 4.

1980s: **Theorem** (*van Geemen / Donagi*).
If we do this for all / for just one double cover and write down all the resulting Schottky-Jung relations (using the full ideal of $Th(\mathcal{A}_{g-1}^{2,4})$), Jacobians will be an irreducible component of the solution set.

+: We get explicit algebraic equations for theta constants.
In fact (*Schottky, Farkas, Rauch, Igusa, Mumford*) we do get the one defining equation in genus 4.

−: We do not really know $Th(\mathcal{A}_{g-1}^{2,4})$ entirely (though we do know many elements of the ideal).
This is only a weak solution, i.e. up to extra components.
Boundary degeneration of Pryms is hard (*Alexeev, Birkenhake, Hulek*).

1970s: *Gunning, Fay, Welters*

For a Jacobian the Kummer image $K(Jac) \subset \mathbf{P}^{2^g-1}$ has many (a 4d family of) trisecant lines.

1970s: *Gunning, Fay, Welters*
For a Jacobian the Kummer image $K(Jac) \subset \mathbf{P}^{2^g-1}$ has many (a 4d family of) trisecant lines.

Theorem
If for some τ the image $K(A_\tau)$ has a family of trisecants, then τ is the period matrix of a Jacobian.
Conjecture. If we know there is one trisecant, τ is already a Jacobian.

1970s: *Gunning, Fay, Welters*
For a Jacobian the Kummer image $K(Jac) \subset \mathbf{P}^{2^g-1}$ has many (a 4d family of) trisecant lines.

Theorem
If for some τ the image $K(A_\tau)$ has a family of trisecants, then τ is the period matrix of a Jacobian.
Conjecture. If we know there is one trisecant, τ is already a Jacobian.

$+$: Get a strong solution (no extra components).

1970s: *Gunning, Fay, Welters*
For a Jacobian the Kummer image $K(Jac) \subset \mathbf{P}^{2^g-1}$ has many (a 4d family of) trisecant lines.

Theorem
If for some τ the image $K(A_\tau)$ has a family of trisecants, then τ is the period matrix of a Jacobian.
Conjecture. If we know there is one trisecant, τ is already a Jacobian.

+: Get a strong solution (no extra components).
−: Need to have a curve in the abelian variety to start with.
The parameters of the trisecant(s) enter in the equations, i.e. we do not directly get algebraic equations for theta constants.

1980s: *Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...*
KP integrable equation as a degenerate trisecant

1980s: *Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...*
KP integrable equation as a degenerate trisecant
Theorem
τ is the period matrix of a Jacobian if and only if $\exists u, v, w \in \mathbf{C}^g, c \in \mathbf{C}$ such that

$$u^4\partial^2\Theta[\varepsilon](\tau,0)+(v^2-uw)\partial\Theta[\varepsilon](\tau,0)+c\Theta[\varepsilon](\tau,0)=0 \quad \forall\varepsilon\in(\mathbf{Z}/2\mathbf{Z})^g.$$

1980s: *Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...*

KP integrable equation as a degenerate trisecant

Theorem

τ is the period matrix of a Jacobian if and only if $\exists u, v, w \in \mathbf{C}^g, c \in \mathbf{C}$ such that

$$u^4\partial^2\Theta[\varepsilon](\tau,0)+(v^2-uw)\partial\Theta[\varepsilon](\tau,0)+c\Theta[\varepsilon](\tau,0)=0 \quad \forall\varepsilon\in(\mathbf{Z}/2\mathbf{Z})^g.$$

$+$: Strong solution — no extra components.

1980s: *Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...*
KP integrable equation as a degenerate trisecant
Theorem
τ is the period matrix of a Jacobian if and only if $\exists u, v, w \in \mathbf{C}^g, c \in \mathbf{C}$ such that

$$u^4\partial^2\Theta[\varepsilon](\tau,0)+(v^2-uw)\partial\Theta[\varepsilon](\tau,0)+c\Theta[\varepsilon](\tau,0) = 0 \quad \forall\varepsilon \in (\mathbf{Z}/2\mathbf{Z})^g.$$

+: Strong solution — no extra components.
−: There are extra parameters (can be eliminated by using effective Nullstellensatz).
This is a differential equation for theta constants, while we are looking for algebraic relations.
Problems with modular invariance.

1990s: *Buser, Sarnak, Lazarsfeld, Nakamaye, Bauer*

Theorem The Seshadri constant for a generic Jacobian is much smaller than for a generic p.p. abelian variety.

1990s: *Buser, Sarnak, Lazarsfeld, Nakamaye, Bauer*

Theorem The Seshadri constant for a generic Jacobian is much smaller than for a generic p.p. abelian variety.

Theorem The shortest period of a generic Jacobian is much shorter than for a generic p.p. abelian variety.

1990s: *Buser, Sarnak, Lazarsfeld, Nakamaye, Bauer*

Theorem The Seshadri constant for a generic Jacobian is much smaller than for a generic p.p. abelian variety.

Theorem The shortest period of a generic Jacobian is much shorter than for a generic p.p. abelian variety.

$+$: Gives an actual doable way to tell that some abelian varieties are *not* Jacobians.

1990s: *Buser, Sarnak, Lazarsfeld, Nakamaye, Bauer*

Theorem The Seshadri constant for a generic Jacobian is much smaller than for a generic p.p. abelian variety.

Theorem The shortest period of a generic Jacobian is much shorter than for a generic p.p. abelian variety.

+: Gives an actual doable way to tell that some abelian varieties are *not* Jacobians.

−: Does not possibly give a way to show that some given abelian variety *is* a Jacobian.

Other approaches:

Andreotti-Mayer: Singularities of the theta divisor

Mumford, Kempf, Muñoz Porras: Geometry of Gauss maps for Jacobians.

Kempf, Ries: Double translation surfaces

G. Farkas: Slopes of modular forms

...

Effectively obtaining the algebraic solution

Theorem 1 (*G.*)

$$a) \qquad \deg Th(\mathcal{A}_g^{2,4}) = N_g \left\langle (\lambda/2)^{\frac{g(g+1)}{2}} \right\rangle_{\overline{\mathcal{A}_g}}$$

$$b) \qquad \deg Th(\mathcal{J}_g^{2,4}) = N_g \left\langle (\lambda/2)^{3g-3} \right\rangle_{\overline{\mathcal{M}_g}}$$

where $\langle \ldots \rangle$ denote the intersection numbers of cohomology classes, and λ is the first Chern class of the Hodge bundle, the bundle of abelian differentials.

Non-proof.

The degree of a subvariety $X \subset \mathbf{P}^{2^g-1}$ of dimension d is the integral of the top power of the Fubini-Study curvature form over it, $\deg X = \int\limits_X \omega_{FS}^d$.

Pull back Fubini-Study to $\mathcal{A}_g^{2,4}$ and $\mathcal{M}_g^{2,4}$ by Th^* and $(Th \circ J)^*$.

Non-proof.

The degree of a subvariety $X \subset \mathbf{P}^{2^g-1}$ of dimension d is the integral of the top power of the Fubini-Study curvature form over it, $\deg X = \int\limits_X \omega_{FS}^d$.

Pull back Fubini-Study to $\mathcal{A}_g^{2,4}$ and $\mathcal{M}_g^{2,4}$ by Th^* and $(Th \circ J)^*$.

Use invariance of ω_{FS} under the level change to push the computation to $\mathcal{A}_g$ and $\mathcal{M}_g$.

Use the fact that the Chern class of the theta bundle is one half of the Chern class of the Hodge bundle (up to torsion).

Non-proof.

The degree of a subvariety $X \subset \mathbf{P}^{2^g-1}$ of dimension d is the integral of the top power of the Fubini-Study curvature form over it, $\deg X = \int\limits_X \omega_{FS}^d$.

Pull back Fubini-Study to $\mathcal{A}_g^{2,4}$ and $\mathcal{M}_g^{2,4}$ by Th^* and $(Th \circ J)^*$.

Use invariance of ω_{FS} under the level change to push the computation to $\mathcal{A}_g$ and $\mathcal{M}_g$.

Use the fact that the Chern class of the theta bundle is one half of the Chern class of the Hodge bundle (up to torsion).

Major difficulty:

Everything blows up at the boundary and we need to extend things there carefully.

Analytic and algebraic intersection numbers for currents may not agree.

The resulting degrees are

g	$\deg Th(\mathcal{J}_g^{2,4})$	$\deg Th(\mathcal{A}_g^{2,4})$
1	1	1
2	1	1
3	16	16
4	208896	13056
5	282654670848	1234714624
6	23303354757572198400	25653961176383488
7	87534047502300588892024209408	197972857997555419746140160

The resulting degrees are

g	$\deg Th(\mathcal{J}_g^{2,4})$	$\deg Th(\mathcal{A}_g^{2,4})$
1	1	1
2	1	1
3	16	16
4	208896	13056
5	282654670848	1234714624
6	23303354757572198400	25653961176383488
7	87534047502300588892024209408	197972857997555419746140160

Corollary. $Th(\mathcal{J}_g^{2,4})$ is not a complete intersection in $Th(\mathcal{A}_g^{2,4})$ for $g = 5, 6, 7$.

Theorem 2.

$$a) \deg Th(\mathcal{A}_g^{2,4}) = N_g (-2)^{-g(g+1)/2} \left(\frac{g(g+1)}{2}\right)! \prod_{k=1}^{g} \frac{\zeta(1-2k)}{2\left((2k-1)!!\right)}$$

Theorem 2.

$$a)\ \deg Th(\mathcal{A}_g^{2,4}) = N_g(-2)^{-g(g+1)/2}\left(\frac{g(g+1)}{2}\right)!\prod_{k=1}^{g}\frac{\zeta(1-2k)}{2\,((2k-1)!!)}$$

$$b)\ \deg Th(\mathcal{J}_g^{2,4}) < C^g\, 2^{2g^2} \quad \text{for some explicit constant } C$$

Theorem 2.

$$a)\, \deg Th(\mathcal{A}_g^{2,4}) = N_g(-2)^{-g(g+1)/2} \left(\frac{g(g+1)}{2}\right)! \prod_{k=1}^{g} \frac{\zeta(1-2k)}{2\,((2k-1)!!)}$$

$$b)\;\; \deg Th(\mathcal{J}_g^{2,4}) < C^g\, 2^{2g^2} \quad \text{for some explicit constant } C$$

Proof. a) follows from *Hirzebruch-Mumford*'s proportionality principle for $\mathcal{A}_g$ and its compact dual

Theorem 2.

$a)\,\deg Th(\mathcal{A}_g^{2,4}) = N_g(-2)^{-g(g+1)/2}\left(\frac{g(g+1)}{2}\right)!\prod_{k=1}^{g}\frac{\zeta(1-2k)}{2\,((2k-1)!!)}$

$b)$ $\deg Th(\mathcal{J}_g^{2,4}) < C^g\,2^{2g^2}$ for some explicit constant C

Proof. a) follows from *Hirzebruch-Mumford*'s proportionality principle for $\mathcal{A}_g$ and its compact dual
b) Use the bound on the Weil-Petersson volume of moduli spaces $vol_{WP}(\mathcal{M}_g) < c^g$ (*G*. 2001), and then Lefschetz index theorem (*Demailly; Yau*):

$$\left\langle \lambda\omega_{WP}^{3g-2}\right\rangle^{3g-3} \geq \left\langle \omega_{WP}^{3g-3}\right\rangle^{3g-2}\left\langle \lambda^{3g-3}\right\rangle.$$

Here the L.H.S. is expressible in terms of WP volumes, as well (*Schumacher, Trapani*).

Theorem 3.
The ideals of algebraic equations for $Th(\mathcal{A}_g^{2,4})$ and for $Th(\mathcal{J}_g^{2,4})$ can be obtained effectively, i.e. there is a finite algorithm that can be applied to get the generators for these ideals.

Theorem 3.
The ideals of algebraic equations for $Th(\mathcal{A}_g^{2,4})$ and for $Th(\mathcal{J}_g^{2,4})$ can be obtained effectively, i.e. there is a finite algorithm that can be applied to get the generators for these ideals.

Proof. For $\mathcal{A}_g$, expand theta constants in power series near some point to order $\deg^2 + 1$ — this gives the germ of $Th(\mathcal{A}_g^{2,4})$. Then any polynomial of degree d in theta constants that vanishes on this germ must vanish along $Th(\mathcal{A}_g^{2,4})$.

Theorem 3.
The ideals of algebraic equations for $Th(\mathcal{A}_g^{2,4})$ and for $Th(\mathcal{J}_g^{2,4})$ can be obtained effectively, i.e. there is a finite algorithm that can be applied to get the generators for these ideals.

Proof. For $\mathcal{A}_g$, expand theta constants in power series near some point to order deg$^2+1$ — this gives the germ of $Th(\mathcal{A}_g^{2,4})$. Then any polynomial of degree d in theta constants that vanishes on this germ must vanish along $Th(\mathcal{A}_g^{2,4})$.
For $\mathcal{J}_g$, first use effective Nullstellensatz to eliminate the parameters in the KP, and then expand the KP equation in Taylor series at some point as well.

Theorem 3.
The ideals of algebraic equations for $Th(\mathcal{A}_g^{2,4})$ and for $Th(\mathcal{J}_g^{2,4})$ can be obtained effectively, i.e. there is a finite algorithm that can be applied to get the generators for these ideals.

Proof. For $\mathcal{A}_g$, expand theta constants in power series near some point to order deg$^2+1$ — this gives the germ of $Th(\mathcal{A}_g^{2,4})$. Then any polynomial of degree d in theta constants that vanishes on this germ must vanish along $Th(\mathcal{A}_g^{2,4})$.
For $\mathcal{J}_g$, first use effective Nullstellensatz to eliminate the parameters in the KP, and then expand the KP equation in Taylor series at some point as well.

Difficulty: The degrees are **HUGE!**

So how do we get the actual algebraic equations?

Motivation:

Addition properties for functions.

Suppose $f : \mathbf{C}^n \to \mathbf{C}$ is such that $f(x)f(y) = f(x+y)\ \forall x, y$. Then f is the exponent $f(x) = \exp(a \cdot x)$ for some $a \in \mathbf{C}^n$.

Motivation:

Addition properties for functions.

Suppose $f : \mathbf{C}^n \to \mathbf{C}$ is such that $f(x)f(y) = f(x+y)\ \forall x, y$. Then f is the exponent $f(x) = \exp(a \cdot x)$ for some $a \in \mathbf{C}^n$.

What if we ask for $g(x+y)f(x)f(y) = 1$? Still the same answer.

Motivation:

Addition properties for functions.

Suppose $f : \mathbf{C}^n \to \mathbf{C}$ is such that $f(x)f(y) = f(x+y)\ \forall x, y$. Then f is the exponent $f(x) = \exp(a \cdot x)$ for some $a \in \mathbf{C}^n$.

What if we ask for $g(x+y)f(x)f(y) = 1$? Still the same answer.

What if we take more than one function, and ask for

$$\sum_{i=1}^{m} g_i(x+y) f_i(x) f_i(y) = 0\ \forall x, y$$

???

Theorem (*Buchstaber, Krichever*).
For any $\tau \in \mathcal{J}_g$, any $x, y \in \mathbf{C}^g$, and any

$$A_0, \ldots, A_{g+1} \in C \subset Jac(C) = A_\tau$$

the following addition property holds:

$$(*) \qquad 0 = \sum_{i=0}^{g+1} c_i(x+y)\theta(A_i + x)\theta(A_i + y),$$

where furthermore c_i can be written explicitly in terms of theta functions.

Theorem (*Buchstaber, Krichever*).
For any $\tau \in \mathcal{J}_g$, any $x, y \in \mathbf{C}^g$, and any

$$A_0, \ldots, A_{g+1} \in C \subset Jac(C) = A_\tau$$

the following addition property holds:

$$(*) \qquad 0 = \sum_{i=0}^{g+1} c_i(x+y)\theta(A_i + x)\theta(A_i + y),$$

where furthermore c_i can be written explicitly in terms of theta functions.

Conjecture (*Buchstaber, Krichever*).
If for some $\tau \in \mathcal{H}_g$ and some $A_i \in \mathbf{C}^g$ the above equation (*) is satisfied for all $x, y \in \mathbf{C}^g$, then $\tau \in \mathcal{J}_g$ and $A_i \in C \subset A_\tau$.

Theorem (*Gunning*).
For any Jacobian the Kummer variety admits a $(2g+2)$-dimensional family of $(g+2)$-secant g-planes. More precisely, $\forall A_0, \ldots A_{g+1} \in C \subset Jac(C)$ and $\forall z \in \mathbf{C}^g$ the $g+2$ points $K(A_i + z)$ inside $\mathbf{P}^{2^g-1}$ are collinear.

Theorem (*Gunning*).
For any Jacobian the Kummer variety admits a $(2g+2)$-dimensional family of $(g+2)$-secant g-planes. More precisely, $\forall A_0, \dots A_{g+1} \in C \subset Jac(C)$ and $\forall z \in \mathbf{C}^g$ the $g+2$ points $K(A_i + z)$ inside $\mathbf{P}^{2^g-1}$ are collinear.

Theorem 4 (*G.*)
Gunning's theorem is equivalent to Buchstaber-Krichever's addition property.

Theorem (*Gunning*).
For any Jacobian the Kummer variety admits a $(2g+2)$-dimensional family of $(g+2)$-secant g-planes. More precisely, $\forall A_0, \dots A_{g+1} \in C \subset Jac(C)$ and $\forall z \in \mathbf{C}^g$ the $g+2$ points $K(A_i + z)$ inside $\mathbf{P}^{2^g-1}$ are collinear.

Theorem 4 (*G.*)
Gunning's theorem is equivalent to Buchstaber-Krichever's addition property.

Theorem 5.
The Buchstaber-Krichever conjecture holds *under some additional assumption of general position*, i.e. both their condition and Gunning's theorem characterize Jacobians and solve the Schottky problem.

Thus we get another solution to the Schottky problem.

Thus we get another solution to the Schottky problem.

Theorem 6.
If for some irreducible $\tau \in \mathcal{H}_g$, $A_i \in \mathbf{C}^g$ with Riemann constant R the following is satisfied for all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ and for all $z \in \mathbf{C}^g$, then $\tau \in \mathcal{J}_g$ and A_i are some points on the curve:

Thus we get another solution to the Schottky problem.

Theorem 6.
If for some irreducible $\tau \in \mathcal{H}_g$, $A_i \in \mathbf{C}^g$ with Riemann constant R the following is satisfied for all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ and for all $z \in \mathbf{C}^g$, then $\tau \in \mathcal{J}_g$ and A_i are some points on the curve:

$$0 = \sum_{\varepsilon \in (\mathbf{Z}/2\mathbf{Z})^g} \Theta[\varepsilon](A_0 + z + R)\Theta[\varepsilon](z)\Theta[\sigma](A_{g+1} + z + R)$$

$$-\Theta[\varepsilon](A_{g+1} + z + R)\Theta[\varepsilon](z)\Theta[\sigma](A_0 + z + R)$$

$$+ \sum_{k=1}^{g} \frac{\theta(2A_{g+1} + R)}{\theta(2A_k + R)} \Theta[\varepsilon](A_0 + z + R)\Theta[\varepsilon](A_{g+1} + z - A_k)\Theta[\sigma](A_k + z + R)$$

unless all the coefficients in front of $\Theta[\sigma](A_i + z + R)$ *are identically zero*

We can also characterize hyperelliptic Jacobians.

Theorem 7.

Let e_i be the unit vector in the i'th dimension, and let $s_i := \sum_{j=1}^{i} e_i$.

Then if for some $\tau \in \mathcal{H}_g$, $A_i \in \mathbf{C}^g$ and all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ and all $z \in \mathbf{C}^g$ the following is satisfied, then τ is a *hyperelliptic* Jacobian and A_i are points on the curve:

We can also characterize hyperelliptic Jacobians.

Theorem 7.

Let e_i be the unit vector in the i'th dimension, and let $s_i := \sum_{j=1}^{i} e_i$.

Then if for some $\tau \in \mathcal{H}_g$, $A_i \in \mathbf{C}^g$ and all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ and all $z \in \mathbf{C}^g$ the following is satisfied, then τ is a *hyperelliptic* Jacobian and A_i are points on the curve:

$$\sum_{\varepsilon} \Theta[\varepsilon](z)\Theta[\varepsilon](z)\Theta[\sigma](z)$$

$$= \sum_{\varepsilon} \sum_{k=1}^{g} (-1)^{(\varepsilon+\sigma,e_k)} \Theta[\varepsilon](z)\Theta[\varepsilon + s_{k-1}]\Theta[\sigma + s_{k-1}](z)$$

$$+ \sum_{\varepsilon} \Theta[\varepsilon](z)\Theta[\varepsilon + s_g](z)\Theta[\sigma + s_g](z),$$

unless all the coefficients in front of $\Theta[\sigma + s_i](z)$ *are identically zero*