Effective algebraic Schottky problem

Samuel Grushevsky
Princeton University

Plan:

1) What is Schottky problem / why is it hard and why is it useful?

2) Notations and definitions

3) History of the problem, and different approaches to solving it

4) Effective algebraic solution

5) (*towards*) Explicit equations for theta functions

Schottky problem is the following question:

which principally polarized abelian varieties are Jacobians of curves?

It relates two important moduli/classification spaces, and would allow one to relate results in algebraic geometry to number theory and modular forms.

A good understanding of the answer could perhaps lead to a better geometric understanding of abelian varieties starting from curves, or allow one to relate the cohomology and other geometric properties of the two moduli spaces.

Notations:

$\mathcal{M}_g$ — moduli space of Riemann surfaces of genus g
$\mathcal{A}_g$ — moduli space of principally polarized abelian varieties
$\mathcal{H}_g$ — Siegel upper half-space for dimension g

$$J : \mathcal{M}_g \to \mathcal{A}_g \qquad \mathcal{J}_g := J(\mathcal{M}_g) \qquad \mathcal{A}_g = \mathcal{H}_g / \operatorname{Sp}(2g, \mathbf{Z})$$

g	$\dim \mathcal{M}_g$		$\dim \mathcal{A}_g$	
1	1	$=$	1	
2	3	$=$	3	
3	6	$=$	6	
4	9	$+1 =$	10	Schottky′s original equation
5	12	$<$	15	Partial geometric results
g	$3g-3$	$<<$	$\frac{g(g+1)}{2}$	???

Theta functions with characteristics

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z) := \sum_{n \in \mathbf{Z}^g} \exp \Big[\big(n + \varepsilon/2, \tau(n+\varepsilon/2)\big) + 2\big(n + \varepsilon/2, z + \delta/2\big) \Big]$$

and theta functions of the second order

$$\Theta[\varepsilon](\tau, z) := \theta \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} (2\tau, 2z)$$

are related via Riemann's bilinear addition theorem

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (z) \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (w) = \sum_{\sigma \in (\mathbf{Z}/2\mathbf{Z})^g} (-1)^{(\delta,\sigma)} \Theta[\sigma + \varepsilon] \left(\frac{z+w}{2} \right) \Theta[\sigma] \left(\frac{z-w}{2} \right)$$

The level subgroups of the modular group are

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \,\middle|\, \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \bmod 2n \right\}.$$

These are normal of $\mathrm{Sp}(2g, \mathbf{Z})$ and their index was computed by *Igusa.* In particular

$$N_g := \# \left(\mathrm{Sp}(2g, \mathbf{Z}) / \Gamma_g(2,4) \right) = 2^{g^2+2g} \prod_{k=1}^{g} (2^{2k} - 1) \sim 2^{2g^2}$$

We denote $\mathcal{A}_g^{2,4} := \mathcal{H}_g/\Gamma_g(2,4)$ and similarly define $\mathcal{M}_g^{2,4}$.

As functions of z for a fixed τ, the functions $\Theta[\varepsilon](\tau,z)$ define the Kummer map of $A_\tau := \mathbf{C}^g/\mathbf{Z}^g + \tau\mathbf{Z}^g$, i.e.

$$\begin{array}{llcc} K: & \mathcal{A}_\tau/\pm 1 & \to & \mathbf{P}^{2^g-1} \\ & z & \to & \left\{\Theta[\varepsilon](\tau,z)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

Theta constants $\Theta[\varepsilon](\tau,0)$ are modular with respect to $\Gamma(2,4)$ of weight one half and thus define the map

$$\begin{array}{llcc} Th: & \mathcal{A}_g^{2,4} & \to & \mathbf{P}^{2^g-1} \\ & \tau & \to & \left\{\Theta[\varepsilon](\tau,0)\right\}_{\text{all}\,\varepsilon}, \end{array}$$

which is known (*R. Salvati Manni*) to be generically injective, and is always at most finite-to-one.

Algebraic Schottky problem:

describe $Th(J(\mathcal{M}_g^{2,4})) \subset Th(\mathcal{A}_g^{2,4})$

History of the problem

1880s: *Schottky* (+1900s *Jung* + perhaps *Riemann* earlier)
Take $\tilde{C} \to C$ an unramified double cover. Let τ be the period matrix of C and let π be the period matrix of the Prym. Then

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}^2 (\pi, 0) = \theta \begin{bmatrix} 0\,\varepsilon \\ 0\,\delta \end{bmatrix} (\tau, 0) \theta \begin{bmatrix} 0\,\varepsilon \\ 1\,\delta \end{bmatrix} (\tau, 0)$$

Using this (together with Riemann's addition theorem) allows us to get some equations for $Th(\mathcal{J}_g^{2,4})$ from equations for $Th(\mathcal{A}_{g-1}^{2,4})$.

1960s: *H. Farkas and Rauch*
prove the validity of this approach, and show that some non-trivial equations result.

1980s: **Theorem** (*van Geemen / Donagi*).
If we do this for all / for just one double cover and write down all the resulting Schottky-Jung relations (using the full ideal of $Th(\mathcal{A}_{g-1}^{2,4})$), Jacobians will be an irreducible component of the solution set.

+: We get explicit algebraic equations for theta constants.
In fact (*Schottky, Farkas, Rauch, Igusa, Mumford*) we do get the one defining equation in genus 4.

−: We do not really know $Th(\mathcal{A}_{g-1}^{2,4})$ entirely (though we do know many elements of the ideal).
This is only a weak solution, i.e. up to extra components.
Boundary degeneration of Pryms is hard (*Alexeev, Birkenhake, Hulek*).

1970s: *Gunning, Fay, Welters*
For a Jacobian the Kummer image $K(Jac) \subset \mathbf{P}^{2^g-1}$ has many (a 4d family of) trisecant lines.

Theorem
If for some τ the image $K(A_\tau)$ has a family of trisecants, then τ is the period matrix of a Jacobian.
Conjecture. If we know there is one trisecant, τ is already a Jacobian.

+: Get a strong solution (no extra components).
−: Need to have a curve in the abelian variety to start with.
The parameters of the trisecant(s) enter in the equations, i.e. we do not directly get algebraic equations for theta constants.

1980s: *Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...*
KP integrable equation as a degenerate trisecant
Theorem
τ is the period matrix of a Jacobian if and only if $\exists u, v, w \in \mathbf{C}^g, c \in \mathbf{C}$ such that

$$u^4\partial^2\Theta[\varepsilon](\tau,0) + (v^2 - uw)\partial\Theta[\varepsilon](\tau,0) + c\Theta[\varepsilon](\tau,0) = 0 \quad \forall\varepsilon \in (\mathbf{Z}/2\mathbf{Z})^g.$$

+: Strong solution — no extra components.
−: There are extra parameters (can be eliminated by using effective Nullstellensatz).
This is a differential equation for theta constants, while we are looking for algebraic relations.
Problems with modular invariance.

1990s: *Buser, Sarnak, Lazarsfeld, Nakamaye, Bauer*

Theorem The Seshadri constant for a generic Jacobian is much smaller than for a generic p.p. abelian variety.

Theorem The shortest period of a generic Jacobian is much shorter than for a generic p.p. abelian variety.

+: Gives an actual doable way to tell that some abelian varieties are *not* Jacobians.

−: Does not possibly give a way to show that some given abelian variety *is* a Jacobian.

Other approaches:

Andreotti-Mayer: Singularities of the theta divisor

Mumford, Kempf, Muñoz Porras: Geometry of Gauss maps for Jacobians.

Kempf, Ries: Double translation surfaces

G. Farkas: Slopes of modular forms

...

Effectively obtaining the algebraic solution

Theorem 1 (*G.*)

$$a) \qquad \deg Th(\mathcal{A}_g^{2,4}) = N_g \left\langle (\lambda/2)^{\frac{g(g+1)}{2}} \right\rangle_{\overline{\mathcal{A}_g}}$$

$$b) \qquad \deg Th(\mathcal{J}_g^{2,4}) = N_g \left\langle (\lambda/2)^{3g-3} \right\rangle_{\overline{\mathcal{M}_g}}$$

where $\langle \ldots \rangle$ denote the intersection numbers of cohomology classes, and λ is the first Chern class of the Hodge bundle, the bundle of abelian differentials.

Non-proof.
The degree of a subvariety $X \subset \mathbf{P}^{2^g-1}$ of dimension d is the integral of the top power of the Fubini-Study curvature form over it, $\deg X = \int_X \omega_{FS}^d$.

Pull back Fubini-Study to $\mathcal{A}_g^{2,4}$ and $\mathcal{M}_g^{2,4}$ by Th^* and $(Th \circ J)^*$.
Use invariance of ω_{FS} under the level change to push the computation to $\mathcal{A}_g$

and $\mathcal{M}_g$.
Use the fact that the Chern class of the theta bundle is one half of the Chern class of the Hodge bundle (up to torsion).
Major difficulty:
Everything blows up at the boundary and we need to extend things there carefully.
Analytic and algebraic intersection numbers for currents may not agree.

The resulting degrees are

g	$\deg Th(\mathcal{J}_g^{2,4})$	$\deg Th(\mathcal{A}_g^{2,4})$
1	1	1
2	1	1
3	16	16
4	208896	13056
5	282654670848	1234714624
6	23303354757572198400	25653961176383488
7	87534047502300588892024209408	197972857997555419746140160

Corollary. $Th(\mathcal{J}_g^{2,4})$ is not a complete intersection in $Th(\mathcal{A}_g^{2,4})$ for $g = 5, 6, 7$.

Theorem 2.

$$a)\ \deg Th(\mathcal{A}_g^{2,4}) = N_g(-2)^{-g(g+1)/2}\left(\frac{g(g+1)}{2}\right)!\prod_{k=1}^{g}\frac{\zeta(1-2k)}{2\left((2k-1)!!\right)}$$

$$b)\ \deg Th(\mathcal{J}_g^{2,4}) < C^g\, 2^{2g^2} \quad \text{for some explicit constant } C$$

Proof. a) follows from *van der Geer*'s computation of the top self-intersection number of λ on $\mathcal{A}_g$
b) Use the bound on the Weil-Petersson volume of moduli spaces $vol_{WP}(\mathcal{M}_g) < c^g$ (*G.* 2001), and then Lefschetz index theorem (*Demailly; Yau*):

$$\left\langle \lambda\omega_{WP}^{3g-2}\right\rangle^{3g-3} \geq \left\langle \omega_{WP}^{3g-3}\right\rangle^{3g-2}\left\langle \lambda^{3g-3}\right\rangle.$$

Here the L.H.S. is expressible in terms of WP volumes, as well (*Schumacher, Trapani*).

Theorem 3.
The ideals of algebraic equations for $Th(\mathcal{A}_g^{2,4})$ and for $Th(\mathcal{J}_g^{2,4})$ can be obtained effectively, i.e. there is a finite algorithm that can be applied to get the generators for these ideals.

Proof. For $\mathcal{A}_g$, expand theta constants in power series near some point to order $\deg^2+1$ — this gives the germ of $Th(\mathcal{A}_g^{2,4})$. Then any polynomial of degree d in theta constants that vanishes on this germ must vanish along $Th(\mathcal{A}_g^{2,4})$.
For $\mathcal{J}_g$, first use effective Nullstellensatz to eliminate the parameters in the KP, and then expand the KP equation in Taylor series at some point as well.

Difficulty: The degrees are **HUGE!**

So how do we get the actual algebraic equations?

Motivation:
Addition properties for functions.

Suppose $f : \mathbf{C}^n \to \mathbf{C}$ is such that $f(x)f(y) = f(x+y)\ \forall x, y$. Then f is the exponent $f(x) = \exp(a \cdot x)$ for some $a \in \mathbf{C}^n$.

What if we ask for $g(x+y)f(x)f(y) = 1$? Still the same answer.

What if we take more than one function, and ask for

$$\sum_{i=1}^{m} g_i(x+y)f_i(x)f_i(y) = 0\ \forall x, y$$

???

Theorem (*Buchstaber, Krichever*).
For any $\tau \in \mathcal{J}_g$, any $x, y \in \mathbf{C}^g$, and any

$$A_0, \ldots, A_{g+1} \in C \subset Jac(C) = A_\tau$$

the following addition property holds:

$$(*) \qquad 0 = \sum_{i=0}^{g+1} c_i(x+y)\theta(A_i+x)\theta(A_i+y),$$

where furthermore c_i can be written explicitly in terms of theta functions.

Conjecture (*Buchstaber, Krichever*).
If for some $\tau \in \mathcal{H}_g$ and some $A_i \in \mathbf{C}^g$ the above equation (*) is satisfied for all $x, y \in \mathbf{C}^g$, then $\tau \in \mathcal{J}_g$ and $A_i \in C \subset A_\tau$.

Theorem (*Gunning*).
For any Jacobian the Kummer variety admits a $(2g+2)$-dimensional family of $g+2$-secant g-planes. More precisely, $\forall A_0, \ldots A_{g+1} \in C \subset Jac(C)$ and $\forall z \in \mathbf{C}^g$ the $g+2$ points $K(A_i+z)$ inside $\mathbf{P}^{2^g-1}$ are collinear.

Theorem 4 (*G.*)
Gunning's theorem is equivalent to Buchstaber-Krichever's addition property.

Theorem 5.
The Buchstaber-Krichever conjecture holds *under some additional assumption of general position*, i.e. both their condition and Gunning's theorem characterize Jacobians and solve the Schottky problem.

Thus we get another solution to the Schottky problem.

Theorem 6.
If for some irreducible $\tau \in \mathcal{H}_g$, $A_i \in \mathbf{C}^g$ with Riemann constant R the following is satisfied for all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ and for all $z \in \mathbf{C}^g$, then $\tau \in \mathcal{J}_g$ and A_i are some points on the curve:

$$0 = \sum_{\varepsilon \in (\mathbf{Z}/2\mathbf{Z})^g} \Theta[\varepsilon](A_0+z+R)\Theta[\varepsilon](z)\Theta[\sigma](A_{g+1}+z+R)$$

$$-\Theta[\varepsilon](A_{g+1}+z+R)\Theta[\varepsilon](z)\Theta[\sigma](A_0+z+R)$$

$$+\sum_{k=1}^{g} \frac{\theta(2A_{g+1}+R)}{\theta(2A_k+R)}\Theta[\varepsilon](A_0+z+R)\Theta[\varepsilon](A_{g+1}+z-A_k)\Theta[\sigma](A_k+z+R)$$

unless all the coefficients in front of $\Theta[\sigma](A_i + z + R)$ *are identically zero*

We can also characterize hyperelliptic Jacobians.

Theorem 7.
Let e_i be the unit vector in the i'th dimension, and let $s_i := \sum_{j=1}^{i} e_i$.
Then if for some $\tau \in \mathcal{H}_g,\ A_i \in \mathbf{C}^g$ and all $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ and all $z \in \mathbf{C}^g$ the following is satisfied, then τ is a *hyperelliptic* Jacobian and A_i are points on the curve:

$$\sum_{\varepsilon} \Theta[\varepsilon](z)\Theta[\varepsilon](z)\Theta[\sigma](z)$$

$$= \sum_{\varepsilon} \sum_{k=1}^{g} (-1)^{(\varepsilon+\sigma, e_k)} \Theta[\varepsilon](z)\Theta[\varepsilon + s_{k-1}]\Theta[\sigma + s_{k-1}](z)$$

$$+ \sum_{\varepsilon} \Theta[\varepsilon](z)\Theta[\varepsilon + s_g](z)\Theta[\sigma + s_g](z),$$

unless all the coefficients in front of $\Theta[\sigma + s_i](z)$ *are identically zero*
