The Schottky Problem

Samuel Grushevsky

Princeton University

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$\mathcal{A}_g$	moduli space of g-dimensional abelian varieties (A, Θ) (complex principally polarized)
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Describe/characterize $\mathcal{J}_g \subset \mathcal{A}_g$.

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- (Super)string scattering amplitudes [D'Hoker-Phong], ...

Dimension counts

g	$\dim \mathcal{M}_g$		$\dim \mathcal{A}_g$	
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2	3	$=$	3	$\mathcal{M}_g = \mathcal{A}_g^{\textit{indecomposable}}$
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g	$3g-3$	$+\frac{(g-3)(g-2)}{2} =$	$\frac{g(g+1)}{2}$	"weak" solutions (up to extra components)

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Claim: $\mathcal{A}_g = \mathcal{H}_g / \mathrm{Sp}(2g,\mathbb{Z})$.

Definition

A modular form of weight k with respect to $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$ is a function $F : \mathcal{H}_g \to \mathbb{C}$ such that

$$F(\gamma\circ\tau) = \det(C\tau + D)^k F(\tau) \qquad \forall\gamma\in\Gamma, \forall\tau\in\mathcal{H}_g$$

Definition

For $\varepsilon, \delta \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g$ the theta function with characteristic ε, δ is

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau, z) := \sum_{N\in\mathbb{Z}^g} \exp\left[\pi i(N+\varepsilon, \tau(N+\varepsilon)) + 2\pi i(N+\varepsilon, z+\delta)\right]$$

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- For $n = 2$, $\theta_m(\tau, z)$ is even/odd in z depending on whether $4\varepsilon \cdot \delta$ is even/odd. For m odd $\theta_m(\tau, 0) \equiv 0$.

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- Theta functions of the second order generate $H^0(A_\tau, 2\Theta)$.

- Theta constants $\theta_m(\tau, 0)$ are modular forms of weight $1/2$ for a certain finite index normal subgroup $\Gamma(2n, 4n) \subset \mathrm{Sp}(2g, \mathbb{Z})$.

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Theorem (Igusa, Mumford, Salvati Manni)

For any $n \geq 2$ theta constants embed

$$\mathcal{A}_g(2n, 4n) := \mathcal{H}_g/\Gamma(2n, 4n) \hookrightarrow \mathbb{P}^{n^{2g}-1}$$

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- Theta constants of the second order define a generically injective $Th : \mathcal{A}_g(2, 4) \to \mathbb{P}^{2^g-1}$ (conjecturally an embedding).

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Classical Riemann-Schottky problem

Write the defining equations for

$$\overline{Th(\mathcal{J}_g(2, 4))} \subset \overline{Th(\mathcal{A}_g(2, 4))} \subset \mathbb{P}^{2^g-1}.$$

g	$\deg Th(\mathcal{J}_g(2,4))$		$\deg Th(\mathcal{A}_g(2,4))$
1	1	$=$	1
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Theorem (Schottky, Igusa)

The defining equation for $\mathcal{J}_4 \subset \mathcal{A}_4$ is

$$F_4(\tau) := 2^4 \sum_{\varepsilon,\delta\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)^{16} - \left(\sum_{\varepsilon,\delta\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} \theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)^{8}\right)^2 .$$

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Exercise. Check if there exist $\mathcal{A}_4$-geodesics lying in $\overline{\mathcal{M}_4}$.

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Open Problem

Construct all geodesics for the metric on $\mathcal{A}_4$ contained in $\overline{\mathcal{M}_4}$.

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Physics conjecture (Belavin, Knizhnik, D'Hoker-Phong, . . .)

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vanishes on $\mathcal{J}_g$ for any g (this is true for $g \leq 4$).

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Theorem (G.-Salvati Manni)

This conjecture is false for any $g \geq 5$.

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This conjecture is false for any $g \geq 5$.
In fact the zero locus of F_5 on $\mathcal{M}_5$ is the divisor of trigonal curves.

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These are the top self-intersection numbers of $\lambda_1/2$ on $\mathcal{M}_g$ and $\mathcal{A}_g$ times the degree of $\mathcal{A}_g(2,4) \to \mathcal{A}_g$.   (G., using Faber's algorithm)

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Corollary

$Th(\mathcal{J}_g(2,4)) \subset Th(\mathcal{A}_g(2,4))$ is not a complete intersection for $g = 5, 6, 7$. (previously proven by Faber)

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These are the top self-intersection numbers of $\lambda_1/2$ on $\mathcal{M}_g$ and $\mathcal{A}_g$ times the degree of $\mathcal{A}_g(2,4) \to \mathcal{A}_g$. (G., using Faber's algorithm)

Corollary

$Th(\mathcal{J}_g(2,4)) \subset Th(\mathcal{A}_g(2,4))$ *is not a complete intersection for* $g = 5, 6, 7$. *(previously proven by Faber)*

Challenge

Write at least one modular form vanishing on $\mathcal{J}_5$.

g	deg $Th(\mathcal{J}_g(2,4))$	deg $Th(\mathcal{A}_g(2,4))$
1	1	1
2	1	1
3	16	16
4	208896	13056
5	282654670848	1234714624
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Write at least one (nice/invariant) modular form vanishing on $\mathcal{J}_5$.

The hyperelliptic Schottky problem

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Theorem (Mumford, Poor)

For any g there exist sets of characteristics $S_1, \ldots, S_N \subset \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ such that $\tau \in \mathcal{A}_g$ is the period matrix of a hyperelliptic Jacobian ($\tau \in Hyp_g$) if and only if for some $1 \leq i \leq N$

$$\forall m \qquad \{\theta_m(\tau) = 0 \Longleftrightarrow m \in S_i\}$$

Schottky-Jung approach (H. Farkas-Rauch)

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Definition

The Prym variety for an étale double cover $\tilde{C} \to C$ of $C \in \mathcal{M}_g$ (given by a point $\eta \in Jac(C)[2] \setminus \{0\}$) is

$$Prym(C, \eta) := Ker_0(Jac(\tilde{C}) \to Jac(C))$$

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Theorem (Schottky-Jung, Farkas-Rauch proportionality)

Let τ be the period matrix of C and let π be the period matrix of the Prym (for the simplest choice of η). Then

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\pi)^2 = const\,\theta\begin{bmatrix}0\,\varepsilon\\ 0\,\delta\end{bmatrix}(\tau)\cdot\theta\begin{bmatrix}0\,\varepsilon\\ 1\,\delta\end{bmatrix}(\tau)\quad \forall\varepsilon,\delta\in\frac{1}{2}\mathbb{Z}^{g-1}/\mathbb{Z}^{g-1}$$

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Using this allows us to get some equations for $Th(\mathcal{J}_g(2,4))$ from equations for $Th(\mathcal{A}_{g-1}(2,4))$.

Theorem (van Geemen)

The locus $\mathcal{J}_g$ *is an irreducible component of the Schottky-Jung locus — the locus obtained by taking the ideal of equations defining* $Th(\mathcal{A}_{g-1}(2,4))$ *and applying the proportionality for all double covers* η.

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$\mathcal{J}_5$ is equal to the Schottky-Jung locus in genus 5.

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- The locus of intermediate Jacobians of cubic threefolds is contained in the "big" (if we take just one η) Schottky-Jung locus in genus 5.
- For $g \geq 7$, $\overline{\mathcal{P}_{g-1}} \subsetneq \mathcal{A}_{g-1}$, so may have more equations $\Rightarrow$ need to solve the Prym Schottky problem if the above is not enough.

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  Boundary degeneration of Pryms is hard
  [Alexeev-Birkenhake-Hulek]

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Theorem (G.-Salvati Manni, Smith-Varley + Debarre)

$$(\mathcal{J}_g \cap \theta_{\rm null}) \subset \theta^3_{\rm null} \subset \theta^{g-1}_{\rm null} \subset (\theta_{\rm null} \cap N_0') \subset \operatorname{Sing} N_0$$

More questions on N_k

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Question

Is it possible that $N_k = N_{k+1}$ for some k?

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- Thus the conjecture is true for $g \leq 5$ ($\overline{\mathcal{P}_5} = \mathcal{A}_5$)

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  Only a weak solution (at least so far) in higher genera.

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Theorem (Welters)

If $\exists C \subset A$ of homology class $2\frac{\Theta^{g-1}}{(g-1)!}$, then A is a Prym (or a degeneration, technical details, ...)

Curves of small homology class

For Jacobians have the Abel-Jacobi curve $C \hookrightarrow Jac(C)$.

Theorem (Matsusaka-Ran)

If $\exists C \subset A$ of "minimal" class $\frac{\Theta^{g-1}}{(g-1)!}$, then $A = Jac(C)$

For Pryms the Abel-Prym curve $\tilde{C} \hookrightarrow Jac(\tilde{C}) \to Prym(\tilde{C} \to C)$.

Theorem (Welters)

If $\exists C \subset A$ of homology class $2\frac{\Theta^{g-1}}{(g-1)!}$, then A is a Prym (or a degeneration, technical details, . . .)

Here we start with a curve and solve an easier version of Schottky:
Given $C \subset A$, is $A = Jac(C)$?

Geometry of the Kummer variety

Definition

The Kummer variety is the image of

$$Kum := |2\Theta| : A_\tau / \pm 1 \hookrightarrow \mathbb{P}^{2^g-1}$$
$$z \longrightarrow \left\{\Theta[\varepsilon](\tau, z)\right\}_{\text{all } \varepsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g},$$

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Trisecant formula (`Fay, Gunning`)

$\forall p, p_1, p_2, p_3 \in C \subset Jac(C) = Pic^0(C)$ the following are collinear:

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This is a solution to the Schottky problem, already given a curve.

“Getting rid” of the points of secancy

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“Multi”secant formula (Gunning)

For any $1 \leq k \leq g$ and for any $p_1, \dots, p_{k+2}, q_1, \dots, q_k \in C \subset Jac(C)$ the $k+2$ points

$$Kum(2p_j + \sum_{i=1}^{k} q_i - \sum_{i=1}^{k+2} p_i), \qquad j = 1 \dots k+2$$

are linearly dependent.

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Conjecture (Buchstaber-Krichever) Theorem (G., Pareschi-Popa)

Given $A \in \mathcal{A}_g^{indec}$ and $p_1, \dots, p_{g+2} \in A$ *in general position*, if

$$\forall z \in A \qquad \{Kum(2p_i + z)\}_{i=1\dots g+2} \subset \mathbb{P}^{2^g-1}$$

are linearly dependent, then $A \in \mathcal{J}_g$.

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  implies that A is a Jacobian.

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So what would be the Prym analog of the trisecant conjecture?

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Challenge

Use these characterizations to approach Coleman's conjecture, or solve the Torelli problem for Pryms (period map generically injective — conjecturally the non-injectivity is due only to the tetragonal construction), or . . .

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$+$ Gives a way to tell that some abelian varieties are *not* Jacobians.
$-$ Does not possibly give a way to show that a given abelian variety *is* a Jacobian, or does it?

Can characterize Hyp_g by the value of their Seshadri constant, if the Γ_{00} conjecture holds [Debarre, Lazarsfeld]

Γ_{00} conjecture

Definition

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- Holds for a generic abelian variety for $g = 5$ or $g \geq 14$. [Beauville-Debarre-Donagi-van der Geer]

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Idea (Muñoz-Porras)

If Γ_{00} conjecture holds, then $\mathcal{J}_g$ =small Schottky-Jung locus (methods to prove this by degenerating to the boundary).

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Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...

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- Krichever showed that the obstruction for extending one flex to a family of flexes vanishes in Taylor series.

- Kadomtsev-Petviashvili (KP) equation is the condition for the existence of a 1-jet of a family of degenerate trisecants (flex lines of the Kummer).
- KP hierarchy of PDEs is the hierarchy of the conditions for the existence of n-jets of a family of flex lines, for each $n \in \mathbb{N}$.
- The existence of an n-jet of a family of flexes for any n gives a formal one-dimensional family of flexes.
- Such a formal family comes from an actual geometric family.
- Thus if the KP hierarchy is satisfied by the theta function, the abelian variety is a Jacobian.
- Shiota proved that the KP equation suffices to recover the hierarchy.
- Krichever showed that the obstruction for extending one flex to a family of flexes vanishes in Taylor series.
- For Pryms, G.-Krichever needed a new hierarchy, etc.