MAT 326 problem set due December 15

All sheaves in this problem set are sheaves of abelian groups (or rings, if the ring structure is implied) over some compact topological space X.

Problem 1.

a) Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ is the collection of sets $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$ a presheaf? Is it a sheaf?

Thinking of $\mathcal{F}$ and $\mathcal{G}$ as sheaves of sections of continuous maps $\pi : \tilde{\mathcal{F}} \to X$ and $\pi' : \tilde{\mathcal{G}} \to X$, define the fiberwise product by

$$\widetilde{\mathcal{F} \times_X \mathcal{G}} := \{v, w | v \in \tilde{\mathcal{F}}, w \in \tilde{\mathcal{G}}, \pi(v) = \pi'(w)\},$$

which of course maps to X. Let then $\mathcal{F} \times_X \mathcal{G}$ be the corresponding sheaf of sections.

b) Let $\mathcal{F} \otimes \mathcal{G}$ be the appropriately defined (to be a sheaf!) tensor product. Prove the following universality property: if $\alpha : \mathcal{F} \times_X \mathcal{G} \to \mathcal{H}$ is a sheaf homomorphism that is bilinear on each stalk, then there exists $\beta : \mathcal{F} \otimes \mathcal{G} \to H$ such that $\alpha = \beta \circ p$, where $p : \mathcal{F} \times_X \mathcal{G} \to \mathcal{F} \otimes \mathcal{G}$ is the natural projection.

Problem 2.

a) Show that over the base $X = \mathbb{C}$ the sequence of sheaves

$$0 \longrightarrow \mathbb{C} \xrightarrow{i} \mathcal{O} \xrightarrow{d} \mathcal{O} \longrightarrow 0,$$

where $\mathbb{C}$ denotes the constant sheaf with fiber $\mathbb{C}$, $\mathcal{O}$ is the sheaf of germs of holomorphic function, i is the inclusion as constant functions, and d is the derivative, is exact.

b) For which open sets $U \subset \mathbb{C}$ is the corresponding sequence

$$0 \longrightarrow \mathbb{C}(U) \xrightarrow{i} \mathcal{O}(U) \xrightarrow{d} \mathcal{O}(U) \longrightarrow 0,$$

exact (i.e when is d surjective here)?

Problem 3.

Let X_1 and X_2 be two closed sets, and let $\mathcal{F}$ be a sheaf on $X := X_1 \cup X_2$.

a) Prove that the sequence $0 \to \mathcal{F} \to \mathcal{F}|_{X_1} \oplus \mathcal{F}|_{X_2} \to \mathcal{F}|_{X_1 \cap X_2} \to 0$ with the maps appropriately defined is exact.

b) Does the sheaf Mayer-Vietoris theorem hold then? Do you need any assumptions on $\mathcal{F}$?

Problem 4.

Show that the cohomology group $H^1(\mathbb{C}^*, \mathbb{C})$ is non-zero (here $\mathbb{C}^*$ denotes the non-zero complex numbers, and we are talking about the cohomology with coefficients in the constant sheaf).