Research description

Samuel Grushevsky

My research interests lie in the area of moduli theory of curves and abelian varieties. Under supervision of Professor Yum-Tong Siu, and inspired by effective techniques in complex algebraic geometry, I have performed research on effective Schottky problem, moduli spaces of Riemann surfaces, a.k.a. algebraic curves ($\mathcal{M}_g$ for closed surfaces of genus g) and theta functions, and am planning to continue the studies in these areas as well as in related areas of mathematical physics, complex algebraic geometry and some aspects of modular forms in several variables over $\mathbb{C}$. My thesis will be entitled "Differential and algebraic equations for the Jacobian locus."

The Riemann-Schottky (or simply Schottky) problem, first posed by Schottky in 1880s, is the problem of characterizing Jacobians among principally polarized abelian varieties (ppavs), i.e. describing the image of $\mathcal{M}_g$ in $\mathcal{A}_g$ by the Jacobian embedding $J : [X] \mapsto [J(X)]$, where $J(X)$ denotes the Jacobian of a Riemann surface X, and $\mathcal{A}_g$ stands for the moduli space of ppavs. The problem has been extensively studied ever since, and numerous solutions have been obtained. However, none of these solutions are effective: given an explicit ppav, it is impossible to apply them to determine whether it is a Jacobian or not. The purpose of my thesis research is to make the solution effective, so that given an explicit ppav, there is a process to establish whether it is a Jacobian or not.

To solve the Schottky problem we first need to introduce coordinates on $\mathcal{A}_g$, and then attempt to describe $\mathcal{J}_g := J(\mathcal{M}_g)$ in these coordinates. To get such coordinates, for any ppav A we can define 2^g theta functions of the second order on A: these are sections of some line bundle (usually denoted 2θ) on A, and are given by the following formula:

$$\Theta[\varepsilon](z,\tau) := \sum_{m\in\mathbb{Z}^g} e^{2\pi i((m+\varepsilon/2,\tau(m+\varepsilon/2))+(m+\varepsilon/2,z))},$$

where τ is an element of Siegel upper half-space (i.e. a symmetric complex $g\times g$ matrix with positive-definite imaginary part) corresponding to the abelian variety A, $z\in\mathbb{C}^g$ and $\varepsilon\in\mathbb{Z}_2^g$ is a vector of g zeros and ones used to label theta functions. These theta functions of the second order linearly generate the space of sections of 2θ over $\mathbb{C}$.

It is known that the theta constants $\Theta[\varepsilon](\tau) := \Theta[\varepsilon](0,\tau)$ embed the moduli space of ppavs into a complex projective space of dimension 2^g-1

(denote this map by $\theta_2 : \mathcal{A}_g \to \mathbb{P}^{2^g-1}$), and that the image $\theta_2(\mathcal{J}_g)$ is an algebraic subvariety in $\mathbb{P}^{2^g-1}$. The equations defining the image $\theta_2(\mathcal{A}_g)$ are not known explicitly, and in fact finding them would provide a solution to the Schottky problem. I have proved the following result about this variety:

Theorem ([Gr2]). *The algebraic degree of $\theta_2(\mathcal{J}_g)$ is less than $c^g 2^{g^2} g^{2g}$ for some explicit constant c.*

Conjecturally this bound gives the right order of growth for the degree, and to prove it, I am currently working on obtaining a lower bound, which constitutes a part of my further research goals.

To compute the degree, first note that the degree of a subvariety in $\mathbb{P}^{2^g-1}$ is equal to the integral over it of the top power of the curvature form of the Fubini-Studi metric. The pullback of this form to $\mathcal{M}_g$ is equal to $\lambda/2$, one half of the first Chern class of the Hodge bundle on $\mathcal{M}_g$ — the g-dimensional vector bundle with fiber over a point representing a Riemann surface being the vector space of all holomorphic differentials on this surface.

Then one would want to say that the integral $\int_{\mathcal{M}_g} \lambda^{3g-3}$ of the top power of the form over $\mathcal{M}_g$ is the same as the cohomological intersection number $\langle \lambda^{3g-3} \rangle_{\overline{\mathcal{M}_g}}$ of the top power of its cohomology class on the (Deligne-Mumford) compactification of the moduli space. However, this may not be true, as the form λ_1 that we are dealing with has singularities on the boundary of $\overline{\mathcal{M}_g}$, and thus the cohomology class is the class of a current, for which the cohomological and analytic intersection numbers may differ.

To deal with this problem, we smooth out the current λ near the boundary $\partial\overline{\mathcal{M}_g}$ to a smooth form μ. Then I use the explicit knowledge of the growth behavior of λ near the boundary to bound the difference $\int \lambda^{3g-3} - \mu^{3g-3}$ by $c^g g^{2g}$ for some constant c. Then the problem is reduced to computing $\int \mu^{3g-3}$, which, since μ is smooth on $\mathcal{M}_g$, is equal to the corresponding cohomological intersection number.

The second cohomology of the moduli space $\overline{\mathcal{M}_g}$ is known (see [Wo]) to be generated by the classes of boundary divisors and the class ω of the Kähler form of the Weil-Petersson metric. As the boundary divisors are isomorphic to moduli spaces of surfaces of lower genera, computing an intersection number involving the boundary divisors can be performed by induction in genus, and it seems that the intersection number hardest to compute is $\langle \omega^{3g-3} \rangle = \int \omega^{3g-3}$ (in [Fab] Faber develops an algorithm to compute all the intersection numbers, which is implementable for low genera).

In paper [Gr1] I have used Penner's [Pe] decorated Teichmüller theory and a combinatorial description of the moduli space of curves to obtain an explicit upper bound on Weil-Petersson volumes of the moduli spaces of punctured Riemann surfaces $\mathcal{M}_{g,n}$ for fixed number of punctures and genus going to infinity:

Theorem ([Gr1]). *For some explicit constant c, and g sufficiently large the following inequality holds for the volumes:* $\mathrm{vol}(\mathcal{M}_{g,n}) \leq c^g g^{2g}$.

Using effective estimates from intersection theory on the moduli space, Schumacher and Trapani [ScTr] have obtained recursive lower bounds on Weil-Petersson volumes, which together with my and Penner's original results (see [Pe]) imply that my upper bound for the volumes has the correct leading order growth.

Applying further effective techniques in intersection theory, together with estimates of self-intersection numbers in terms of intersection number with a nef divisor, and leaning heavily on the results on Weil-Petersson volumes described above, the bound $\langle \mu^{3g-3} \rangle < c^g g^{2g}$, and thus a bound on the algebraic degree of $\theta_2(\mathcal{J}_g)$, is obtained.

The techniques developed in the computations above, and the bounds obtained, allow one in principle to approach the problem of determining which divisors on the compactification $\overline{\mathcal{M}}_g$ are effective. Studying the effective cone of $\overline{\mathcal{M}}_g$ has been an outstanding research problem in the past 20 years, with extensive work performed by Mumford, Harris, Morrison, Moriwaki, Keel and others: see [HaMu], [HaMo], [Mo], [Ke] and references therein. I am planning to apply the effective techniques and the explicit knowledge of certain two-forms on the moduli space to further this study, with a view towards results in intersection and Morse theory of the moduli spaces.

In another approach to Schottky problem, one starts with the Kadomtsev-Petviashvili (KP) equation, which was shown by Shiota [Sh] to provide a solution to Schottky problem: it is a differential equation for theta function, and if the theta function of a ppav satisfies it, the ppav is a Jacobian. However, if given an explicit ppav (say, as a period matrix τ in the Siegel upper half-space), it is not clear how to verify whether it is a Jacobian or not using the KP equation. There are two problems for such a verification. First, the KP equation involves $3g+1$ unknown parameters, which need to be eliminated. Secondly, verifying the KP amounts to verifying the vanishing of a certain expression in theta constants and their derivatives. Theta constants and their derivatives are given by infinite power series in τ, and thus verifying vanishing of such an expression requires checking vanishing of each of infinitely many terms in power series. As derivatives of theta constants are not modular forms, we cannot a priori bound the degree of the terms for which we need to check vanishing (such results for modular forms are known).

To deal with the first problem, I have derived a system of second-order parameterless differential equations for theta constants of any ppav.

Theorem ([Gr4]). *Considering the first derivatives of theta constants with respect to τ as rank one tensors, second derivatives as rank two tensors, and multiplying everything tensorly, the following equations hold for any ppav (not necessarily a Jacobian) with a period matrix τ:*

$$\forall\varepsilon \quad \sum_{\sigma\in\mathbb{Z}_2^g}\Theta[\sigma+\varepsilon](\tau)^2\Theta[\sigma](\tau)\partial^2\Theta[\sigma](\tau)=$$

$$=\sum_{\sigma\in\mathbb{Z}_2^g}\Theta[\sigma+\varepsilon](\tau)^2\partial\Theta[\sigma](\tau)\partial\Theta[\sigma](\tau)+2\Theta[\sigma+\varepsilon](\tau)\Theta[\sigma](\tau)\partial\Theta[\sigma+\varepsilon](\tau)\partial\Theta[\sigma](\tau).$$

I showed this system of equations to be equivalent to Ohyama's equations in [Oh] and to all the equations contained in [Zu]. Using these equations to express (non-linearly) second derivatives of theta constants in terms of the first, we can rewrite the KP as a system of first-order differential equations for theta constants with parameters. Using resultants and effective Nullstellensatz (see [EiLa]) we can then eliminate the unknowns to derive a system of non-linear first-order differential equations for theta constants of second order equivalent to the KP, which is thus explicit and yields itself to potential verification for a given period matrix. I plan to exhibit this work as [Gr4] and to compose my thesis of it together with [Gr1], [Gr2], and a review of connections between differential-geometric and intersection-theoretic approaches and techniques involved.

However, the second problem — of how to verify a differential identity for theta constants — remains. Using the results of [Gr2] one can a priori bound the degree of the algebraic equations for the Jacobian locus, and thus stop the term-by-term (in power series expansion in τ) verification of a differential identity for theta constants when the degree of the term is sufficiently high. This gives an effective process for solving the Schottky problem for an explicit ppav:

Theorem ([Gr3]). *Using the bound for the degree of the Jacobian locus and effective Nullstellensatz, the KP equation for the theta function can be effectively rewritten as a finite system of algebraic equations for theta constants.*

Though the procedure used to derive these algebraic equations is effective, it is so complicated that actually applying it, even in low genus, is very hard. I am planning to pursue this idea with the goal of obtaining the explicit algebraic equations between theta constants defining the Jacobian locus in genus five, the lowest genus for which such equations are not known (there are no equations for $g \leq 3$ and the unique equation in genus four was already known to Schottky).

The question of determining differential and algebraic equations for theta constants of general abelian varieties is also still open: knowing such equations would then allow one to write explicitly the equations for theta constants of Jacobians following from Schottky-Jung proportionalities (see [Far] for a review of these ideas). The question itself has a significance for the theory of modular forms of several variables, where one wants to find the differential operators mapping modular forms (or some sets of modular forms) of some weight to modular forms of some other weight. Much work has been done in that direction in dimension one by Shimura [Shi], Resnikoff [Re] and others, and I would like to perform research with the goal of obtaining an effective bound on the dimension of the jet space of modular forms. Such a bound would then enable us to estimate the order of the differential operators involved, and perhaps to obtain explicitly such differential operators.

References

[EiLa] Ein, L., Lazarsfeld, R.: *A geometric effective Nullstellensatz.* Invent. Math. **137** (1999), no. 2, 427–448

[Fab] Faber, C.: *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians.* New trends in algebraic geometry (Warwick, 1996), 93–109, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999

[Far] Farkas, H.: *Schottky-Jung theory.* Theta functions — Bowdoin 1987, 459–483

[Gr1] Grushevsky, S.: *An explicit upper bound for Weil-Petersson volumes of the moduli spaces of punctured Riemann surfaces*, Math. Ann. **321** (2001), 1–13

[Gr2] Grushevsky, S.: *The degree of the Jacobian locus*, submitted

[Gr3] Grushevsky, S.: *Effective algebraic Schottky problem*, submitted

[Gr4] Grushevsky, S.: *Parameterless first-order differential equations characterizing the Jacobian locus*, chapter 4 of Harvard University Ph.D. dissertation, 2002

[HaMo] Harris, J., Morrison, I.: *Slopes of effective divisors on the moduli space of stable curves.* Invent. Math. **99** (1990), no. 2, 321–355

[HaMu] Harris, J., Mumford, D.: *On the Kodaira dimension of the moduli space of curves.* Invent. Math. **67** (1982), 23–86

[Ke] Keel, S.: *Basepoint freeness for nef and big line bundles in positive characteristic.* Ann. of Math. (2) **149** (1999), no. 1, 253–286

[Mo] Moriwaki, A.: *Nef divisors in codimension one on the moduli space of stable curves.* preprint math.AG/0005012

[Oh] Ohyama, Y.: *Differential equations of theta constants in genus two*, Algebraic analysis of singular perturbations (Kyoto 1996), 96–103, translation: preprint of Osaka University, 1996

[Pe] Penner, R.: *Weil-Petersson volumes*, J. Differential Geometry **35** (1992) 559–608

[Re] Resnikoff, H.: *Differential equations for automorphic forms in several complex variables.* Proc. Amer. Math. Soc. **24** (1970) 492–496

[Shi] Shimura, G.: *On the derivatives of theta functions and modular forms.* Duke Math. J. **44** (1977), no. 2, 365– 387.

[Sh] Shiota, T.: *Characterization of Jacobian varieties in terms of soliton equations.* Invent. Math. **83** (1986), no. 2, 333–382.

[ScTr] Schumacher, G. and Trapani, S.: *Estimates of Weil-Petersson volumes via effective divisors*, Commun. Math. Phys. **222** (2001) no. 1, 1–7

[Wo] S. Wolpert *On the homology of the moduli space of stable curves*, Annals of Mathematics **118** (1983), 491–523

[Zu] Zudilin, V.: *Theta constants and differential equations* Mat. Sb. **191** (2000), no. 12, 77–122, translation in *Russian Acad. Sci. Sb. Math.*