Cohomology of compact hyper-Kähler manifolds

and the bound on their second Betti number

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Compact hyper-Kähler manifold

Definition

A compact Kähler manifold $X = (M, I)$ is called hyper-Kähler (HK) if:

- X is simply connected.
- $\exists$ unique holomorphic symplectic 2-form σ on X.
  (up to constant)

Examples

Dimension 2: K3 surface

- K3 surface = compact HK manifold of dimension 2.
- Intensively studied over decades.
- Hodge structure of $H^*(\mathrm{K3}, \mathbb{Q})$ is well understood.

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Example. Hodge diamond of K3:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

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- (Beauville '83) Generalized Kummer variety Kum_n. Dimension $2n$.
- (O'Grady '99 '03) Sporadic examples OG6, OG10. Dimension 6 and 10.

Goal

Ultimate goal

Understand the cohomology $H^*(X, \mathbb{Q})$ of a HK manifold X.

Previous results

Theorem (Göttsche–Soergel '93)

There is an explicit formula for Hodge structures of $H^*(\mathrm{K3}^{[n]}, \mathbb{Q})$ and $H^*(\mathrm{Kum}_n, \mathbb{Q})$.

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Example. Hodge diamond of $\mathrm{K3}^{[2]}$ and Kum_2:

$$\begin{matrix} &&&&1&&&& \\ &&&0&&0&&& \\ &&1&&21&&1&& \\ &0&&0&&0&&0& \\ 1&&21&&232&&21&&1 \\ &0&&0&&0&&0& \\ &&1&&21&&1&& \\ &&&0&&0&&& \\ &&&&1&&&& \end{matrix} \qquad \begin{matrix} &&&&1&&&& \\ &&&0&&0&&& \\ &&1&&5&&1&& \\ &0&&4&&4&&0& \\ 1&&5&&96&&5&&1 \\ &0&&4&&4&&0& \\ &&1&&5&&1&& \\ &&&0&&0&&& \\ &&&&1&&&& \end{matrix}$$

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Theorem (Mongardi–Rapagnetta–Saccà '18, de Cataldo–Rapagnetta–Saccà '19)

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Example. Hodge diamond of OG10 (1st quadrant, no odd cohom):

1					
22	1				
254	22	1			
2299	276	23	1		
16490	2531	276	22	1	
88024	16490	2299	254	22	1

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$H^*(X, \mathbb{Q})$ admits a $\mathfrak{g}$-module structure, where $\mathfrak{g}$ is a Lie algebra.

Here $\mathfrak{g}$-module structure means:

$$H^*(X, \mathbb{Q}) = \bigoplus_{\mu} m_\mu V_\mu$$

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Example. (Markman '03) $H^*(\mathrm{K3}^{[3]}, \mathbb{Q}) = V_{\square\square\square} \oplus V_{\substack{\square\\ \square}}$.

Main Theorem

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- There is an explicit formula for $\mathfrak{g}$-module structures of $H^*(\mathrm{K3}^{[n]}, \mathbb{Q})$ and $H^*(\mathrm{Kum}_n, \mathbb{Q})$.
- $\mathfrak{g}$-module structures for OG6 and OG10 are:

$$H^*(\mathrm{OG6}, \mathbb{Q}) = V_{\ydiagram{3}} \oplus V_{\ydiagram{1,1,1}} \oplus 135\, V_{\ydiagram{1}} \oplus 240\mathbb{Q},$$

$$H^*(\mathrm{OG10}, \mathbb{Q}) = V_{\ydiagram{5}} \oplus V_{\ydiagram{2,2}}.$$

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Example.

- $H^*(\mathrm{K3}^{[4]}, \mathbb{Q}) = V_{\ydiagram{4}} \oplus V_{\ydiagram{2,1}} \oplus V_{\ydiagram{2}} \oplus \mathbb{Q}.$
- $H^*(\mathrm{Kum}_2, \mathbb{Q}) = V_{\ydiagram{2}} \oplus 80\mathbb{Q} \quad (\oplus \text{ odd cohom}).$

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Moral: We know the most refined cohomology structure for all known examples of HK.

Application 1: Degeneration of HK

Conjecture (Nagai '08)

Let $\mathfrak{X}/\Delta$: degenerating family of HK and ν_{2k}: monodromy index on $H^{2k}(X, \mathbb{Q})$. Then in fact,

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Theorem (Green–K–Laza–Robles '19)

Nagai's conjecture $\Leftrightarrow$ Knowing what Young diagrams appear in $H^*(X, \mathbb{Q})$.

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Hence we know what Young diagram appears! $\square$

Example. $H^*(\mathrm{K3}^{[4]}, \mathbb{Q}) = V_{\square\square\square\square} \oplus V_{\substack{\square\square \\ \square}} \oplus V_{\square\square} \oplus \mathbb{Q}$.
Young diagrams $\square\square\square\square$, $\substack{\square\square \\ \square}$, $\square\square$, $\varnothing$.
$\Rightarrow$ Nagai's conjecture holds for $\mathrm{K3}^{[4]}$.

Application 2: Second Betti number of HK

We highly suspect the following (stronger than Nagai's conjecture):

Conjecture (Green–K–Laza–Robles '19)

For all HK,

$$(\text{size of each Young diagrams in } H^*(X,\mathbb{Q})) \leq \frac{1}{2}\dim X.$$

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The conjecture implies:

Theorem (K–Laza '19)

If the Conjecture holds, then $\exists$ explicit bound on b_2 for HK.

$\dim X$	2	4	6	8	10	12	14	≥ 16
$b_2(X) \leq$	22	23	23	24	25	26	27	$2 \dim X - 1$

Thank you!