

Cohomology of compact hyper-Kähler manifolds and the bound on their second Betti number

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Compact hyper-Kähler manifold

Definition

A compact Kähler manifold $X = (M, I)$ is called **hyper-Kähler (HK)** if:

- X is simply connected.
- \exists unique holomorphic symplectic 2-form σ on X .
(up to constant)

Examples

Dimension 2: K3 surface

- K3 surface = compact HK manifold of dimension 2.
- Intensively studied over decades.
- **Hodge structure** of $H^*(K3, \mathbb{Q})$ is well understood.

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Example. Hodge diamond of K3:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

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- (Beauville '83) Generalized Kummer variety Kum_n . Dimension $2n$.
- (O'Grady '99 '03) Sporadic examples OG_6 , OG_{10} . Dimension 6 and 10.

Goal

Ultimate goal

Understand the cohomology $H^*(X, \mathbb{Q})$ of a HK manifold X .

Previous results

Theorem (Göttsche–Soergel '93)

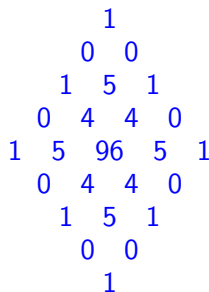
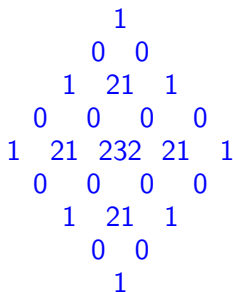
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Example. Hodge diamond of $K3^{[2]}$ and Kum_2 :



Previous results

Theorem (Mongardi–Rapagnetta–Saccà '18,
de Cataldo–Rapagnetta–Saccà '19)

- (MRS '18) Computed **Hodge diamond** of OG6.
- (dCRS '19) Computed **Hodge diamond** of OG10.

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Example. Hodge diamond of OG10 (1st quadrant, no odd cohom):

1					
22	1				
254	22	1			
2299	276	23	1		
16490	2531	276	22	1	
88024	16490	2299	254	22	1

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Theorem (Verbitsky '95, Looijenga–Lunts '97)

$H^*(X, \mathbb{Q})$ admits a **\mathfrak{g} -module** structure, where \mathfrak{g} is a Lie algebra.

Here **\mathfrak{g} -module** structure means:

$$H^*(X, \mathbb{Q}) = \bigoplus_{\mu} m_{\mu} V_{\mu}$$

where μ : Young diagram (collection of boxes).

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Example. (Markman '03) $H^*(K3^{[3]}, \mathbb{Q}) = V_{\square\square\square} \oplus V_{\square}.$

Main Theorem

Main Theorem (Green–K–Laza–Robles '19)

- There is an explicit formula for **g-module** structures of $H^*(K3^{[n]}, \mathbb{Q})$ and $H^*(Kum_n, \mathbb{Q})$.
- **g-module** structures for OG6 and OG10 are:

$$H^*(OG6, \mathbb{Q}) = V_{\square\square\square} \oplus V_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} \oplus 135V_{\square} \oplus 240\mathbb{Q},$$

$$H^*(OG10, \mathbb{Q}) = V_{\square\square\square\square} \oplus V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}.$$

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Example.

- $H^*(K3^{[4]}, \mathbb{Q}) = V_{\square\square\square\square} \oplus V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \oplus V_{\square\square} \oplus \mathbb{Q}.$
- $H^*(Kum_2, \mathbb{Q}) = V_{\square\square} \oplus 80\mathbb{Q} \quad (\oplus \text{ odd cohom}).$

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Moral: We know the most refined cohomology structure for **all known examples** of HK.

Application 1: Degeneration of HK

Conjecture (Nagai '08)

Let \mathfrak{X}/Δ : degenerating family of HK and ν_{2k} : monodromy index on $H^{2k}(X, \mathbb{Q})$. Then in fact,

$$\nu_{2k} = k \cdot \nu_2.$$

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Theorem (Green–K–Laza–Robles '19)

Nagai's conjecture \Leftrightarrow Knowing what Young diagrams appear in $H^*(X, \mathbb{Q})$.

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Young diagrams $\square\square\square\square$, $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, $\square\square$, \emptyset .

\Rightarrow Nagai's conjecture holds for $K3^{[4]}$.

Application 2: Second Betti number of HK

We highly suspect the following (stronger than Nagai's conjecture):

Conjecture (Green–K–Laza–Robles '19)

For all HK,

$$(\text{size of each Young diagrams in } H^*(X, \mathbb{Q})) \leq \frac{1}{2} \dim X.$$

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The conjecture implies:

Theorem (K–Laza '19)

If the Conjecture holds, then \exists explicit bound on b_2 for HK.

$\dim X$	2	4	6	8	10	12	14	≥ 16
$b_2(X) \leq$	22	23	23	24	25	26	27	$2 \dim X - 1$

Thank you!