Measuring Irrationality

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Definition

A *rational map* $X \dashrightarrow Y$ between two varieties is a morphism $U \to V$ where $U \subset X$ and $V \subset Y$ are Zariski-open subsets.

Definition

Two varieties X and Y are *birational* (denoted $X \simeq_{\text{bir.}} Y$) if they are isomorphic on Zariski-open subsets. In other words, there exist rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ whose composition is the identity (as a rational map).

Motivation

The question of whether or not a variety is rational has been studied extensively.

Question

Given a projective variety of dimension n which is not rational, how far away is it from being rational?

When $n = 1$, the natural invariant is the *gonality* of a curve C, which measures the smallest degree of a branched covering

$$C' \to \mathbb{P}^1,$$

where C' is a normalization of C.

$$\mathrm{gon}(C) = 1 \iff C \simeq_{\text{bir.}} \mathbb{P}^1.$$

Measures of Irrationality

One possible invariant in higher dimensions:

Definition

The *degree of irrationality* of a projective variety X of dimension n is

$$\mathrm{irr}(X) = \min \left\{ \delta > 0 \;\middle|\; \begin{array}{c} \exists \text{ degree } \delta \text{ rational covering} \\ X \dashrightarrow \mathbb{P}^n \end{array} \right\}$$

- As we saw before,

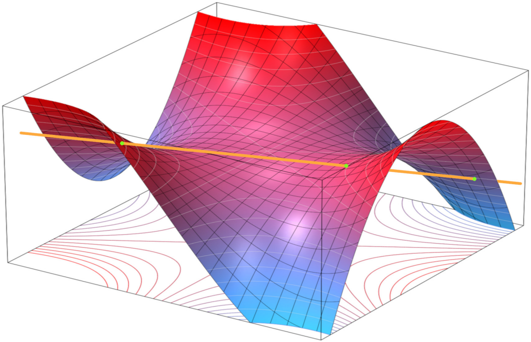
$$\mathrm{irr}(X) = 1 \iff X \simeq_{\text{bir.}} \mathbb{P}^n.$$

- This is a birational invariant.

Example

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension n and degree d. By projecting from any point $p \in X$, we obtain a rational map

$$f : X \dashrightarrow \mathbb{P}^n \quad \text{with} \quad \deg(f) = d - 1.$$



Recent results

Theorem (Bastianelli, De Poi, Ein, Lazarsfeld, Ullery; 2017)

Let $X \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface of dimension n and degree $d \geq 2n+1$. Then

$$\mathrm{irr}(X) = d - 1.$$

Main theme of [BDELU]: the invariants measuring irrationality are related to the positivity of the canonical bundle.

Question

What about when K_X is trivial?

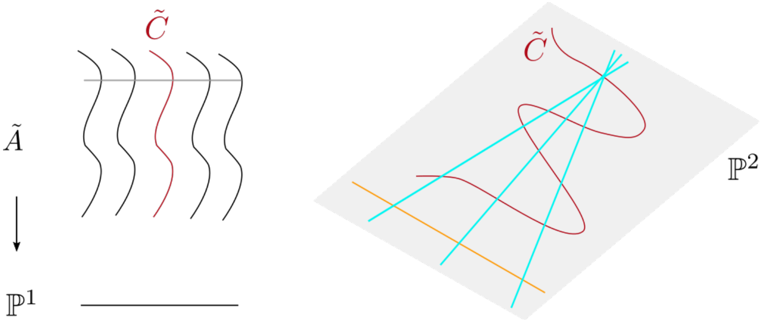
Polarized abelian surfaces

- Let $A = A_d$ be a very general abelian surface carrying a polarization $L = L_d$ of type $(1, d)$.
- Numerically, this means that

$$h^0(A, L) = d, \quad \int c_1(L)^2 = 2d.$$

One implication:
The abelian surface A_d does NOT contain any smooth curves of genus $\leq d$.

- (Alzati-Pirola) It has been proven that $\text{irr}(A) \geq 3$ for any abelian surface.
- (Yoshihara) $\text{irr}(A_2) = 3$.



Polarized abelian surfaces

- An argument of Stapleton [5] showed that there is a positive constant C such that

$$\mathrm{irr}(A_d) \leq C \cdot \sqrt{d}.$$

- It was conjectured in [BDELU] that equality holds asymptotically. In other words,

$$\liminf_{d \to \infty} \mathrm{irr}(A_d) = \infty.$$

Theorem (C.; 2019)

For an abelian surface $A = A_d$ as before, one has

$$\mathrm{irr}(A_d) \leq 4.$$

- The proof involves explicitly constructing 4-to-1 maps

$$A \dashrightarrow \mathbb{P}^2$$

for every degree d.

Fano hypersurfaces

As mentioned earlier, [BDELU] gives sharp bounds for the degree of irrationality of hypersurfaces of large degree ($d \geq 2n + 1$). In particular, these varieties have positive K_X.

Question

What can be shown for hypersurfaces where K_X is negative?

This consists of hypersurfaces $X_{n,d} \subset \mathbb{P}^{n+1}$ of degree d where

$$d \leq n + 1.$$

Let $X_{n,d} \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface of dimension n and degree d.

Theorem (C.-Stapleton; 2019)

If $d \geq n + 1 - \frac{\sqrt{n+2}}{4}$, then

$$\operatorname{irr}(X_{n,d}) \geq \frac{\sqrt{n+2}}{4}.$$

References

[1] Alberto Alzati and Gian Pietro Pirola, On the holomorphic length of a complex projective variety, *Arch. Math.* **59** (1992), 398 – 402.

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[3] Nathan Chen, Degree of irrationality of very general abelian surfaces, Algebra Number Theory **13** (2019), 2191 – 2198.

[4] Nathan Chen and David Stapleton, Fano hypersurfaces with arbitrarily large degrees of irrationality, preprint, arXiv:1908.02803 (2019).

[5] David Stapleton, The degree of irrationality of very general hypersurfaces in some homogeneous spaces, Ph.D. thesis, Stony Brook University, 2017.

[6] Hisao Yoshihara, Degree of irrationality of a product of two elliptic curves, *Proc. Amer. Math. Soc.* **124** (1996), 1371–1375.