The boundary of orbit closures

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Graduate Student Recitals
Stony Brook, March 31

Moduli spaces of differentials

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$\mathcal{H}$ is the moduli space of Abelian differentials up to isomorphism, also called a stratum.

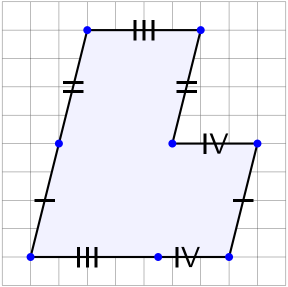
An alternate description of $\mathcal{H}$

$$\mathcal{H} = \{\text{Abelian differentials}\}$$

$$\leftrightarrow$$

$$\{\text{polygons in the plane with parallel sides identified}\}$$

Constructing holomorphic differentials



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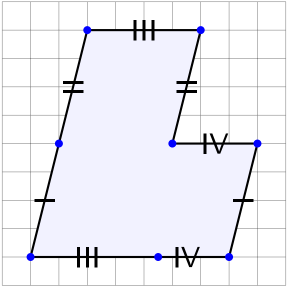

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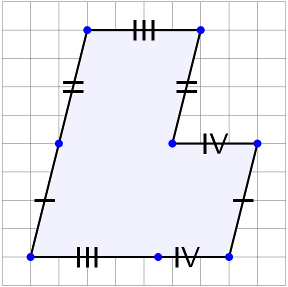

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- $z \mapsto z + c$ is holomorphic $\Rightarrow$ X is a Riemann surface
- $d(z + c) = dz \Rightarrow$
  dz descends to a holomorphic differential on X

Strata as spaces of polygons

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Consequences

- $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{H}$
- $\mathcal{H}$ has distinguished local coordinates given by the complex lengths of the polygon, so called period coordinates.

The $SL(2,\mathbb{R})$-action

- $SL(2,\mathbb{R})$ acts on $\mathbb{C} \simeq \mathbb{R}^2$

The $SL(2, \mathbb{R})$-action

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The $SL(2,\mathbb{R})$-action

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- the action keeps parallel sides parallel

The $\mathrm{SL}(2,\mathbb{R})$-action

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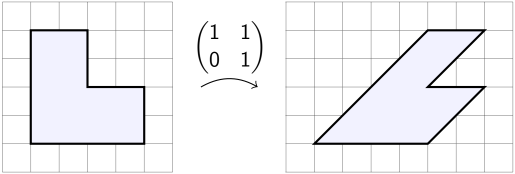

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Fix a point (X, ω) in a stratum and choose a polygonal representation $\mathcal{P}$.

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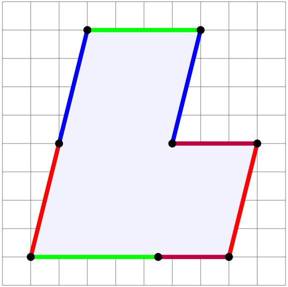

$\mathcal{P}$

Period coordinates

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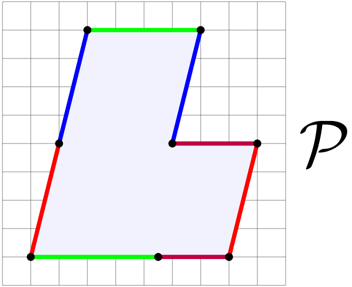

"Fact"

The complex lengths of the edges of $\mathcal{P}$ give local coordinate charts near (X, ω), called period coordinates. Furthermore, the transition functions are linear transformations.

Magic Wand and Algebraicity

Theorem (Eskin-Mirzakhani-Mohammadi 2013, Filip 2013)

Orbit closures $\overline{\mathrm{SL}(2,\mathbb{R})\cdot(X,\omega)}$ *are algebraic varieties which, in period coordinates, are given by linear equations with real coefficients.*

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Orbit closures are never compact.

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Goal

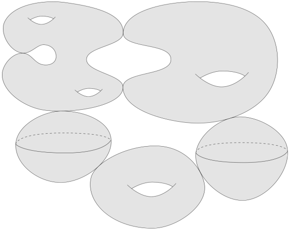
Find a suitable compactification.

Compactifying strata

Theorem (Bainbridge-Chen-Gendron-Grushevsky-Möller 2019)

There exists a compactification Ξ of the stratum $\mathcal{H}$ such that

- Ξ *is smooth*



A nodal Riemann surface

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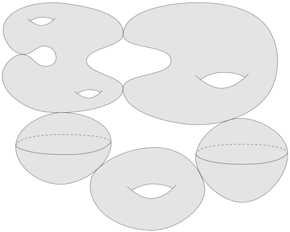

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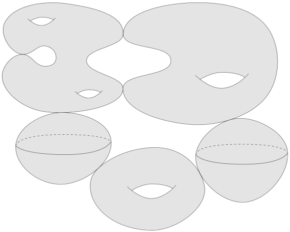
- *the boundary $\partial\Xi = \Xi \setminus \mathcal{H}$ consists of nodal Riemann surfaces equipped with meromorphic differentials*
- *the boundary $\Xi \setminus \mathcal{H}$ has distinguished period coordinates*



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Let $M \subseteq \mathcal{H}$ be an orbit closure for the $\mathrm{SL}(2,\mathbb{R})$-action. Then the boundary $\partial M \subseteq \Xi$ is, locally in the period coordinates of the boundary, given by linear equations with real coefficients.

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Further directions

- Classification of orbit closures