Spring 2017 MAT 536, Complex Analysis
Instructor: Samuel Grushevsky
Homework #2, due in class Wed February 8

Problem 1. (a) State and prove the chain rule for differentiation of *smooth* functions $f : \mathbb{R}^2 \to \mathbb{R}^2$.

(b) Write the result of (a) in terms of coordinates z and $\bar{z}$ instead of x and y.

(c) State and prove the chain rule for differentiation of holomorphic functions.

Problem 2. Prove that the function

$$f(z) := \begin{cases} \overline{z}^2/z, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at the origin, but is not differentiable there.

Problem 3. Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the (open) upper half-plane. Does there exist a holomorphic non-constant function $f : \mathbb{C} \to \mathbb{H}$?

Problem 4. Find a biholomorphism $\Delta \to \mathbb{H}$, where $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.

Problem 5. For any $n \in \mathbb{N}$ prove that $f(z) = z^{1/n}$, suitably defined, is an analytic function $f : \mathbb{H} \to \mathbb{C}$, and compute all of its derivatives at $z = i$.

Problem 6. Recall that (we have not proven this, but still) the group of biholomorphisms of $\mathbb{H}$ is equal to the fractional linear transformations $\mathrm{PSL}_2(\mathbb{R})$. Construct a Riemannian metric (that is, something of the form $f(x, y)dxdy$, please read up on a proper definition) on $\mathbb{H}$ that is invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$, and prove such a metric is unique up to scaling.

(Optional: if you know what curvature is, compute the curvature of this metric. Could you have known the answer beforehand, without a computation?)