Equations for affine invariant manifolds, via degeneration

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BiSTRO seminar
May 31, 2021

- Joint work with *Frederik Benirschke* and *Benjamin Dozier*
- Applies a compactification constructed with *Matt Bainbridge, Dawei Chen, Quentin Gendron, Martin Möller*
- Uses *Frederik*'s thesis

Strata of holomorphicmeromorphic differentials

- $X \in \mathcal{M}_g$ = genus g Riemann surface
- $z_1, \ldots, z_n \in X$ = distinct numbered marked points
- $\omega \in H^0(X, K_X) = H^{1,0}(X, \mathbb{C}) =$ holomorphic$\omega \in H^0(X, K_X + \sum m_i z_i)$ =meromorphic 1-form on X

Definition

For $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0} \in \mathbb{Z}$ the stratum is

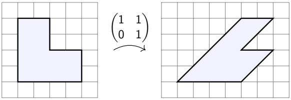
$$\mathcal{H}_{g,n}(\mu) := \{(X, z_1, \ldots, z_n, \omega \neq 0) \colon \operatorname{ord}_{z_i}\omega = m_i\}$$

and ω has no zeroes or poles on $X \setminus \{z_1, \ldots, z_n\}$.

Projectivized stratum $\mathcal{P}_{g,n}(\mu) := \mathcal{H}_{g,n}(\mu)/\mathbb{C}^*$

Period coordinates and $\mathrm{GL}^+(2, \mathbb{R})$ action

- Local coordinates on a *holomorphic* stratum: integrals of ω over a basis of $H_1(X, \{z_1, \ldots, z_n\}; \mathbb{Z}) = H_1(X, \text{Zeroes}; \mathbb{Z})$.
- Local coordinates on a *meromorphic* stratum: integrals of ω over a basis of $H_1(X \setminus \text{Poles}, \text{Zeroes}; \mathbb{Z})$.
- $\mathrm{GL}^+(2, \mathbb{R})$ action on the stratum. In local *period coordinates* $\mathcal{H}_{g,n}(\mu) \simeq \mathbb{C}^N \simeq (\mathbb{R}^2)^{\times N}$, and let $\mathrm{GL}^+(2, \mathbb{R})$ act on $\mathbb{R}^2$. ($N = 2g + n - 1$ for holomorphic, $N = 2g + n - 2$ for meromorphic)



Theorem (Eskin-Mirzakhani-Mohammadi)

For holomorphic strata, orbit closures are locally given in period coordinates by linear equations with real coefficients.

(Linear equations with $\mathbb{R}$ coefficients are preserved by $\mathrm{GL}^+(2, \mathbb{R})$)

Theorem (Filip)

*For holomorphic strata, orbit closures are (quasi-projective) algebraic

Towards classifying $\mathrm{GL}^+(2,\mathbb{R})$ orbit closures

affine invariant manifold := orbit closure in a holomorphic stratum

- Teichmüller curves = closed orbits; map to complex curves in $\mathcal{P}_{g,n}(\mu)$
- Covering constructions
- Upper bounds on the rank of primitive orbit closures (Mirzakhani-Wright, Apisa-Wright, ...)
- Gothic locus and quadrilateral constructions (McMullen-Mukamel-Wright, Eskin-McMullen-Mukamel-Wright)
- Meromorphic strata: ???

Idea:
Study orbit closures via degenerations

Degenerations

- $\mathcal{H}_{g,n}(\mu)$ is not compact: can degenerate the Riemann surface and/or the differential
- $\mathcal{P}_{g,n}(\mu)$ is not compact: can degenerate the Riemann surface
- No orbit closure in $\mathcal{P}_{g,n}(\mu)$ is compact. Can consider

$$\lim_{\lambda\to\infty}\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\circ(X,\omega)$$

- What about $\lim_{\lambda\to\infty}\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}\circ(X,\omega)$?

Moduli of multi-scale differentials [BCG–M]

$\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) = \Xi\overline{\mathcal{M}}_{g,n}(\mu)/\mathbb{C}^*$ is a compactification of $\mathcal{P}_{g,n}(\mu)$ that is algebraic, smooth (as an orbifold), $\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \overline{\mathcal{M}}_{g,n}$, boundary $\partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is a normal crossing divisor.

Points of $\partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ correspond to nodal Riemann surfaces, with their components fully, weakly ordered by "scale" (how fast the volume went to zero), together with a meromorphic differential on each component, plus prong-matchings and conditions.

Upshot

Locally any boundary stratum of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is a product of strata of meromorphic differentials, satisfying some linear conditions on residues.

Why degenerations restrict linear equations

$\mathcal{H}_{g,n}(\mu) \supset M :=$ affine invariant manifold=orbit closure

$F :=$ (local) defining equation for M near $p \in M$

Write $F(X,\omega) = \int_\gamma \omega = 0$ for some $\gamma \in H_1(X, Zeroes; \mathbb{C})$

- Suppose $F(X,\omega) = \int_\alpha \omega - \int_\beta \omega$, where $\alpha \cdot \beta = 1$ are intersecting classes in $H_1(X;\mathbb{Z})$
- Suppose *within* M can pinch α to a node, without pinching anything else crossed by β
- "Near" such a limit point cannot distinguish β from $N\alpha + \beta$, for $N \in \mathbb{Z}$
- So locally could have $\int_\beta \omega = N \int_\alpha \omega$ for any $N \in \mathbb{Z}$
- Infinitely many components, certainly non-algebraic . . .

Vertical and horizontal vanishing cycles

- $M \subset \mathcal{H}_{g,n}(\mu)$; closure $\overline{M} \subset \Xi\overline{\mathcal{M}}_{g,n}(\mu)$.
- Fix $p_0 \in \partial\overline{M}$.
  Fix $\Gamma :=$ dual graph of X_0, with level structure.
- *Horizontal* edges $E^{hor}(\Gamma)$ connect vertices of same level.
  Vertical edges connect vertices of different levels.
- $p_0 \in D_\Gamma :=$ open boundary stratum of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.
  (fixed dual graph, no further degenerations; fixed prong-matching, all locally in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$)
- $\forall p = (X, \omega) \in M$ sufficiently close to p_0 can be obtained by *plumbing* some $q \in D_\Gamma$.
  Nodes e at q are opened up to seams at p, aka vanishing cycles $\lambda_e \in H_1(X; \mathbb{Z})$.

Monodromy argument [*Benirschke*]

Lemma

For any $p = (X, \omega) \in M$ sufficiently close to p_0, let $\{\lambda_e\}_{e \in E(\Gamma)}$ be the collection of all vanishing cycles on X. Then for any defining equation F for M at p, there exist $n_e \in \mathbb{Z}$ such that

$$\sum_e n_e \langle F, \lambda_e \rangle \int_{\lambda_e} \omega = 0$$

is also a defining equation for M at p.

Proof

Let $f : \Delta \to \overline{M}$ map $0 \mapsto p_0$ and $\frac{1}{2} \mapsto p$. Analytically continue coordinates from p along a loop around zero, starting and returning to p, and keep writing the equation F. □

Components of $\partial\overline{M}$

$$\operatorname{codim}_{\Xi\overline{\mathcal{M}}_{g,n}(\mu)} D_\Gamma = (\text{number of levels in } \Gamma \text{ minus } 1) + (\text{number of horizontal nodes})$$

Theorem (BD–)

If $\dim \overline{M} \cap D_\Gamma = \dim M - 1$, then either

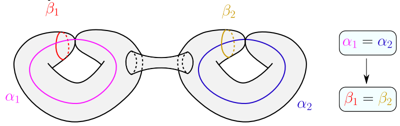
- *Γ has two levels, and no horizontal nodes, or*
- *Γ is all at one level, and periods over any two horizontal vanishing nodes are proportional on M.for any two horizontal vanishing cycles λ_1, λ_2, there is a defining equation for M of the form $c\int_{\lambda_1}\omega = \int_{\lambda_2}\omega$.*

Proportionality of periods over horizontal vanishing cycles

Theorem (BD–)

If two horizontal vanishing cycles λ_1, λ_2 are M-cross-related, (i.e. $\exists F$ a defining equation for M such that $\langle F, \lambda_1\rangle \cdot \langle F, \lambda_2\rangle \neq 0$, F cannot be written as $F \neq F_1 + F_2$ with $\langle F_1, \lambda_2\rangle = \langle F_2, \lambda_1\rangle = 0$ ... or there is a chain of such F's) then there is a defining equation for M of the form $c\int_{\lambda_1}\omega = \int_{\lambda_2}\omega$.

Example:



$$c_1\int_{\alpha_1}\omega + c_2\int_{\alpha_2}\omega + c_3\int_{\alpha_3}\omega = 0$$

(and $\nexists$ other equations crossing a subset of $\beta_1, \beta_2, \beta_3$) implies that periods over $\beta_1, \beta_2, \beta_3$ are *pairwise* proportional.

Minimal holomorphic stratum $\mathcal{H}_{g,1}(2g-2)$

Easier because there are no relative periods. Coordinates: $H_1(X;\mathbb{Z})$

Theorem (BD–)

For $M \subset \mathcal{H}_{g,1}(2g-2)$ affine invariant manifold, let $\{\lambda_e\}_{e\in E^{hor}(\Gamma)}$ be the set of all horizontal vanishing cycles. Then

1. *The space of linear relations among periods over λ_e is generated by pairwise proportionalities $c\int_{\lambda_{e_i}}\omega = \int_{\lambda_{e_j}}\omega$.*
2. *If λ_{e_i} and λ_{e_j} are M-cross-related, then there is a defining equation F_{ij} that crosses only $\lambda_{e_i}, \lambda_{e_j}$ and no other horizontal vanishing cycles.*

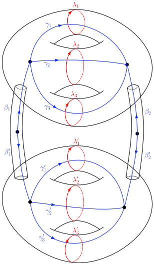
- (1) always holds for divisorial degenerations — here for any D_Γ
- The proof crucially uses the result of *Avila-Eskin-Möller* that $TM \subset H_1(X;\mathbb{Z})$ is symplectic.
- For non-minimal strata, can have complicated relations among λ_e in $H_1(X, Zeroes;\mathbb{Z})$.

Counterexample to generalizing the statement for the minimal holomorphic stratum $\mathcal{H}_{g,1}(2g-2)$

Theorem (BD–)

1. *The space of linear relations among periods over horizontal vanishing cycles λ_e is generated by pairwise proportionalities $c\int_{\lambda_{e_i}}\omega = \int_{\lambda_{e_j}}\omega$.*
2. *If λ_{e_i} and λ_{e_j} are M-cross-related, then there is a defining equation F_{ij} that crosses only $\lambda_{e_i}, \lambda_{e_j}$ and no other horizontal vanishing cycles.*

Counterexample in $\mathcal{H}_{5,8}(1,1,1,1,1,1,1,1)$: 4-branched double covers of $\mathcal{H}_{2,2}(1,1)$



Period equations cutting out M
$\int_{\lambda_1}\omega - \int_{\lambda_1'}\omega = 0$
$\int_{\lambda_2}\omega - \int_{\lambda_2'}\omega = 0$
$\int_{\lambda_3}\omega - \int_{\lambda_3'}\omega = 0$
$\int_{\gamma_1}\omega - \int_{\gamma_1'}\omega = 0$
$\int_{\gamma_2}\omega - \int_{\gamma_2'}\omega = 0$
$\int_{\gamma_3}\omega - \int_{\gamma_3'}\omega = 0$
$\int_{\beta_1}\omega - \int_{\beta_1'}\omega = 0$
$\int_{\beta_2}\omega - \int_{\beta_2'}\omega = 0$

NOTE:

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_{1'} + \lambda_{2'} + \lambda_{3'} = 0 \in H_1(X;\mathbb{Z})$$

Then $2\int_{\lambda_1}\omega + 2\int_{\lambda_2}\omega + 2\int_{\lambda_3}\omega = 0$ holds on M, *but* there are *no pairwise* proportionalities among $\int_{\lambda_i}\omega$.

Linear subvarieties in general

Definition

A linear subvariety in a meromorphic stratum is an algebraic variety locally near any point given by linear equations, with arbitrary complex coefficients.

- Any interesting examples in holomorphic strata?
- In general *not* preserved by the $\mathrm{GL}^+(2,\mathbb{R})$ action.

Theorem (Benirschke)

Any boundary stratum $\overline{M} \cap \partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of any linear subvariety is a product of linear subvarieties for the strata corresponding to the components of the nodal curve.

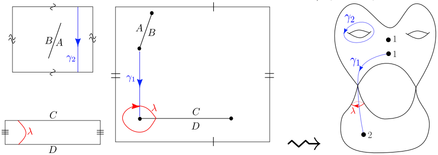
General structural results for linear subvarieties

Theorem (BD–)

1. *For any defining equation F of M, the collection of periods over vertical vanishing cycles that cross a given level i and are crossed by F satisfy a linear relation.*
2. *The space of defining equations of M can be generated by equations that only cross horizontal nodes at one level, and equations that do not cross any horizontal nodes at all.*
3. *Local equations for $\overline{M}$ near p_0 in plumbing coordinates on $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ can be computed explicitly from the local linear defining equations nearby.*
4. *In particular, $\overline{M}$ locally near ∂M looks like a toric variety (possibly non-normal).*

How to apply this

Example: ruling out a linear subvariety in $\mathcal{H}_{3,3}(1,1,2)$:



Then the one equation $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ does *NOT* define an affine invariant manifold, because otherwise must have $\int_\lambda \omega = 0$ as another defining equation.

Cylinder Deformation Theorem [*Wright*]

Definition

Parallel flat cylinders: periods of ω over circumference curves are real multiples of each other.
M-parallel cylinders: remain parallel for all nearby $(X, \omega) \in M$.

Theorem (Wright)

Let $\mathcal{C}$ be a maximal collection of M-parallel cylinders, for some $(X, \omega) \in M$. Then applying $GL^+(2, \mathbb{R})$ to cylinders in $\mathcal{C}$ and leaving the rest of X untouched gives a flat surface also in M.

- So, in a way, the relations on M involving curves on cylinders only involve curves on $M-$parallel cylinders.
- BD– give a new proof, for linear subvarieties of *meromorphic* strata, if all coefficients of defining equations are *real*.
- The theorem is for *smooth* Riemann surfaces. Our proof is *by degeneration* to nodal Riemann surfaces.

Idea of our proof of Cylinder Deformation Theorem

1. To get close to the boundary, apply $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, as $\lambda \to \infty$, to *all of X, not just $\mathcal{C}$*. This stretches cylinders and limits to nodes.
   Q: What do cylinders look like near $\partial \Xi \overline{\mathcal{M}}_{g,n}(\mu)$?
   A: For a sufficiently small neighborhood of a boundary point, all circumference curves of cylinders of sufficiently large modulus come from vanishing horizontal cycles.
2. Write the defining equations for M at $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \circ (X, \omega)$ as sums of equations that don't cross any horizontal vanishing cycles, and equations H crossing some set $\{\lambda_1, \ldots, \lambda_k\}$.
3. The nodes crossed by each H are M-cross-related, so periods over vanishing cycles are pairwise proportional.
4. So all of $\lambda_1, \ldots, \lambda_k$ lie on M-parallel cylinders.
5. So deforming $\lambda_1, \ldots, \lambda_k$ all at once preserves the equation H, and so stays on M. □

Thank you

(and please apply this)