Equations for affine invariant manifolds, via degeneration

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- Joint work with *Frederik Benirschke* and *Benjamin Dozier*

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- Thanks to *Fred* for those pictures that are nice!

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Projectivized stratum $\mathcal{P}_{g,n}(\mu) := \mathcal{H}_{g,n}(\mu)/\mathbb{C}^*$

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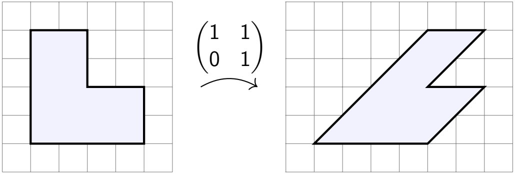

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Theorem (Filip)

For holomorphic strata, orbit closures are (quasi-projective) algebraic varieties.

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Idea:

Study orbit closures via degenerations.

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$$\lim_{\lambda \to \infty} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \circ (X, \omega)$$

- What about $\lim_{\lambda \to \infty} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \circ (X, \omega)$?

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- Infinitely many components, certainly non-algebraic ...

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  (fixed dual graph, no further degenerations; fixed prong-matching, all locally in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$)
- $\forall p = (X, \omega) \in M$ sufficiently close to p_0 can be obtained by *plumbing* some $q \in D_\Gamma$.
  Nodes e are opened up to seams, aka *vanishing cycles* $\lambda_e \in H_1(X, \mathbb{Z})$.

Monodromy argument [*Benirschke*]

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Lemma

For $p_0 \in \partial M \cap D_\Gamma$*, for any* $p = (X, \omega) \in M$ *sufficiently close to* p_0*, let* $\{\lambda_e\}$ *be the collection of all vanishing cycles on* X*.*

Monodromy argument [*Benirschke*]

Lemma

For $p_0 \in \partial M \cap D_\Gamma$*, for any* $p = (X, \omega) \in M$ *sufficiently close to* p_0*, let* $\{\lambda_e\}$ *be the collection of all vanishing cycles on* X*. Then for any defining equation* F *for* M *at* p*, there exist* $n_e \in \mathbb{Z}$ *such that*

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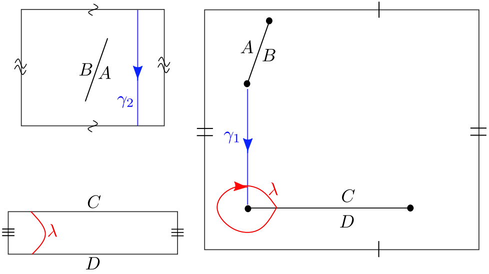

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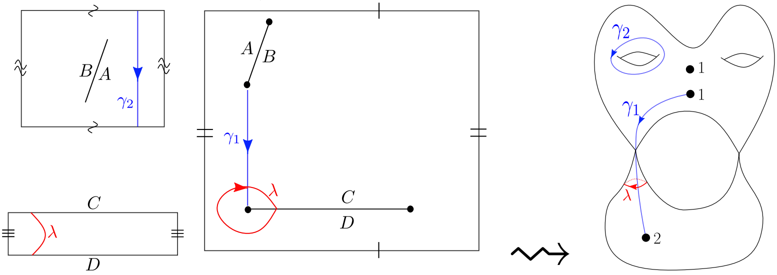

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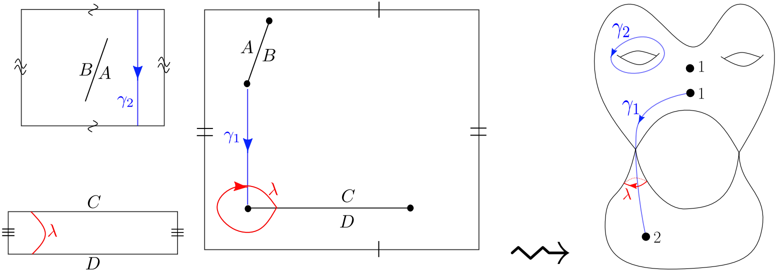

Then the one equation $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ does *NOT* define an affine invariant manifold.

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- The proof crucially uses the result of *Avila-Eskin-Möller* that $TM \subset H_1(X;\mathbb{Z})$ is symplectic
- For non-minimal strata, can have complicated relations among the classes of λ_i in $H_1(X, Zeroes; \mathbb{Z})$

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Theorem (Benirschke)

Any boundary stratum $\overline{M} \cap \partial \Xi \overline{\mathcal{M}}_{g,n}(\mu)$ *of any linear subvariety is a product of linear subvarieties for the strata corresponding to the components of the nodal curve.*

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3. *Local equations for $\overline{\mathcal{M}}$ near p_0 in plumbing coordinates on $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ can be computed explicitly from the local linear defining equations nearby.*
4. *In particular, $\overline{\mathcal{M}}$ locally near ∂M looks like a toric variety (possibly non-normal).*

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- The theorem is for *smooth* Riemann surfaces. Our proof is *by degeneration* to nodal Riemann surfaces.