MAT200, Lecture 1

Midterm II. November 8, 2010

This is a closed notes/ closed book/ electronics off exam.

You are allowed and encouraged to draw pictures to illustrate and motivate your proofs, but your proofs should be formal logical derivations. For your convenience, all the definitions, axioms and theorems are summarized below — you are allowed to use any of them by making a clear reference (please write "by the following axiom/theorem: ..."). Please write a detailed proof for every problem, justifying every statement you make by referring to appropriate axioms or theorems. *Proofs by "it is clear from the picture" will get very little partial credit.*

Please write legibly and cross out anything that you do not want the grader to read.

Each problem is worth 20 points.

Name:

Problem	1	2	3	4	5	Total
Grade						

Geometry. Below is an *informal* summary of the definitions, axioms, and theorems from geometry that you can use.

- **Basic Terms.** The *Euclidean plane* $\mathbb{E}$ is a set, its elements are called points, denoted $A, B, C, \dots$. The *lines*, denoted $\ell, m, n, \dots$ are certain subsets of the plane. The *distance* is a function $\mathbb{E} \times \mathbb{E} \to \mathbb{R}$, denoted by $|AB|$. Unless stated otherwise, in all the statements below all the points are assumed to be distinct.
- Definition 2.1: Lines are called transverse if they are not equal and have a non-empty intersection. They are called parallel otherwise.
- **Incidence Axiom.** There exist two distinct points; for any distinct points A and B there exists a unique line (denoted $\overleftrightarrow{AB}$) containing A and B; for any line ℓ there exists a point of $\mathbb{E}$ not lying on ℓ
- **The Parallel Axiom.** For any point A and any line ℓ there exists a unique line containing A and parallel to ℓ.
- Theorem 2.1: transverse lines meet in exactly one point.
- Theorem 2.2: two lines parallel to a third one are parallel.
- **The Ruler Axiom.** On any line ℓ there exists a coordinate system — a bijection $f : \ell \to \mathbb{R}$ such that $\forall A, B \in \ell, |AB| = |f(A) - f(B)|$.
- Theorem 3.1: $\forall A, B$, there exists a unique coordinate system on $\overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$.
- Definition 3.1: For $A, B, C \in \ell$ we say that B is between A and C if for some coordinate system f on ℓ we have $f(A)$ $f(B) < f(C)$. This is then used to define points on the same and opposite sides of another point on a line (definition 3.3).
- Definition 3.4: The ray $\overrightarrow{AB}$ is the set of all points of $\overleftrightarrow{AB}$ which are on the same side of A as B.
- Theorem 3.4: If $C \in \overrightarrow{AB}$, then $\overrightarrow{AC} = \overrightarrow{AB}$.
- Theorem 3.5: The distance $|AB|$ is always non-negative, and is equal to zero if and only if $A = B$.
- Definition 4.1: For $A, B \notin \ell$, if $AB \cap \ell \neq \emptyset$, we say A and B are on opposite sides of ℓ. Otherwise they are on the same side.
- **Plane Separation Axiom.** For any line ℓ and any three distinct points A, B, C not lying on ℓ: if A and B are on the same side of ℓ, and B and C are on the same side of ℓ, then A and C are on the same side of ℓ; if A and B are on the opposite sides of ℓ, and B and C are on the opposite sides of ℓ, then A and C are on the same side of ℓ.
- Theorem 4.1: Any line cuts the plane into two half-planes.

- Definition 4.5: The interior of a (non-straight) $\angle BAC$ is the set of points D that are on the same side of $\overleftrightarrow{AB}$ as C, and on the same side of $\overleftrightarrow{AC}$ as D.
- **The Protractor Axiom.** The measure of any angle is greater than 0, and less than or equal to π; the measure of any straight angle is π. For any $A \neq B$, any $D \notin \overleftrightarrow{AB}$, and any $\alpha \in \mathbb{R}$ with $0 < \alpha < \pi$, there exists a unique ray $\overrightarrow{AC}$ on the same side of $\overleftrightarrow{AB}$ as D and such that $m\angle BAC = \alpha$. If $\overrightarrow{AD}$ lies in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.
- Theorem 4.2: Monotonicity of angles: if C and D are on the same side of $\overleftrightarrow{AB}$, then $m\angle BAD < m\angle BAC$ if and only if $\overrightarrow{AD}$ lies in the interior of $\angle BAC$.
- Theorem 4.3: (Crossbar theorem) A ray $\overrightarrow{AD}$ is in the interior of a non-straight angle $\angle BAC$ if and only $\overrightarrow{AD} \cap BC \neq \emptyset$.
- Theorem 4.4: The measures of vertical angles are equal; of supplementary — add up to π.
- Definition 5.1: triangles are congruent if corresponding sides have equal lengths, and corresponding angles — equal measures
- **The SAS Congruence Axiom.**
- Theorem 5.1: ASA congruence of triangles.
- Definition 5.5: $\triangle ABC$ is isosceles if $\triangle ABC = \triangle BAC$.
- Theorem 5.2: a triangle is isosceles if an only if the base angles are equal.
- Theorem 5.3: SSS congruence of triangles.
- Theorem 5.7: In $\triangle ABC$, $m\angle A > m\angle B \iff |BC| > |AC|$.
- Theorem 5.9: In any triangle $|AB| + |BC| > |AC|$.
- Theorem 6.2: Two lines are parallel if and only if the alternate interior angles are equal
- Theorem 6.4: For any point A and any line ℓ there exists a unique line n containing ℓ and perpendicular to ℓ.
- Theorem 6.5: The sum of the measures of the angles of a triangle is equal to π.
- Exercise 6.4: The measure of an external angle of a triangle is equal to the sum of the measures of the two other angles.
- Definition 6.1: convex quadrilaterals, parallelograms
- Theorem 6.7: The measures of opposite angles of a parallelogram are equal; the lengths of the opposite sides of a parallelogram are equal.
- Theorem 6.8: If the lengths of opposite sides of a quadrilateral are equal, it is a parallelogram.

Problem 1. Let ℓ and m be transverse lines. For every point $A \in \ell$, prove that there exists a line n containing A, such that n is transverse to ℓ and parallel to m.

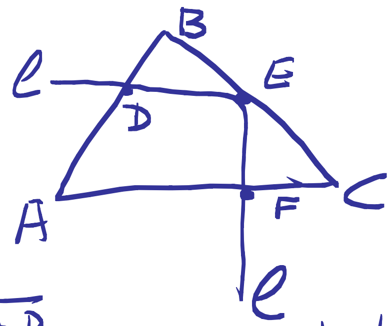
By the parallel axiom, there exists a line n containing A and parallel to m ($\exists\, n \ni A,\ n \parallel m$). Thus we need to show that $n \pitchfork \ell$. By definition, $n \pitchfork \ell$ if $n \cap \ell \neq \emptyset$ and $n \neq \ell$.

Indeed, $A \in n \cap \ell$, so the intersection is non-empty. Note also that n is parallel to m, while ℓ is transverse to m, so certainly $n \neq \ell$.

Problem 2. Prove that a line ℓ cannot intersect all three sides of the triangle $\triangle ABC$, i.e. all three of the segments AB, BC, AC (recall that in our definition the segment does not contain its endpoints, so ℓ does not go through any vertex of the triangle).

Hint: Assume for contradiction that ℓ intersects these segments in points D, E, F, respectively. Are the points D and F on the same or opposite sides of $\overleftrightarrow{BC}$?

Assume that on the line ℓ we have $D-E-F$ (otherwise we can rename A, B, C, and D, E, F), so the picture would be



Since $D \in \overline{AB}$, we must have $A-D-B$, and then A and D lie on the same side of $\overleftrightarrow{BC}$ (since $B = \overleftrightarrow{AB} \cap \overleftrightarrow{BC}$ is not contained in $\overline{AB}$).

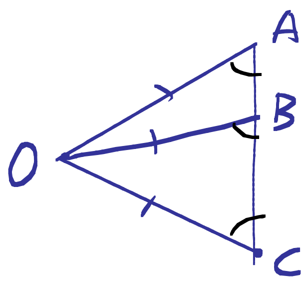
Similarly A and F lie on the same side of $\overleftrightarrow{BC}$.

By Plane Separation Axiom it then follows that D and F lie on the same side of $\overleftrightarrow{BC}$, and so $\overline{DF} \cap \overleftrightarrow{BC} = \emptyset$, but this contradicts $E = \overline{DF} \cap \overleftrightarrow{BC}$.

Problem 3. The *circle* with center $O \in \mathbb{E}$ and radius $r \in \mathbb{R}_+$ is defined to be the set of all points $A \in \mathbb{E}$ such that $|AO| = r$. Prove that a line cannot intersect a circle in more than two points.

Hint: assume for contradiction that a line intersects a circle in at least 3 points, and consider the resulting isosceles triangles.

Assume that A, B, C are some three points of intersection of ℓ with a circle, labeled in such a way that $A-B-C$ on ℓ. By definition of the circle, $|OA| = |OB| = |OC| = r$:



Since $\triangle BOC$ is isosceles, $m\angle OCB = m\angle OBC$. Since $\triangle AOC$ is isosceles, also $m\angle OAC = m\angle OCA$.

But $\angle OBC$ is an exterior angle of $\triangle AOB$, and thus by the exterior angle inequality we must have $m\angle OBC > m\angle OAC$, but we just proved that the measures of these angles are equal (both equal to $m\angle OCA$). Thus we have a contradiction.

(Alternatively, note $m\angle OBA = m\angle OAB$ since $\triangle AOB$ is isosceles. Then $\pi = m\angle OBC + m\angle OBA$, so all angles are $\pi/2$, but then $OA \parallel OB$)

Problem 4. Let $\mathbb{F} = \{A, B, C, D\}$ be the set consisting of 4 elements. Make $\mathbb{F}$ into a "plane" (i.e. list all the subsets of it that you call lines) satisfying the incidence axiom, the parallel axiom, and satisfying the version of the ruler axiom with $\mathbb{R}$ replaced by the set consisting of two elements, 0 and 1 (that is to say that for any line ℓ there exists a bijection $f : \ell \to \{0, 1\}$ such that $|AB| = |f(A) - f(B)|$ for any two points $A, B \in \ell$).

Note that each line here must be in bijection with $\{0,1\}$, and thus must contain exactly two points, distance of 1 apart.

To satisfy the incidence axiom, we must have a line through any two points, so the lines would be

$\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{B,D\}$, $\{C,D\}$.

Any two distinct points must then be a distance of 1 apart, and then the incidence and "ruler" axiom are satisfied. One then checks that the parallel axiom is satisfied, eg. that $\{A,B\} \parallel \{C,D\}$, etc.

(One of the practice midterm problems was actually this construction, and we discussed it in class)

Problem 5. Prove that there does not exist a plane $\mathbb{F}$ satisfying the properties listed in the previous problem, and such that $\mathbb{F}$ consists of 5 distinct points.

Hint: how many points must a line contain? How many lines must there be in $\mathbb{F}$ then?

Let $\mathbb{F} = \{A, B, C, D, E\}$. As in problem 4), by the "ruler" axiom each line must consist of two points, and to satisfy the incidence axiom any set consisting of two points must be a line. But then we have

$\{A, B\} \parallel \{C, D\}$ and also

$\{A, B\} \parallel \{C, E\}$.

Thus there are two lines containing C that are parallel to $\{A, B\}$, and the parallel axiom fails.