MAT200, Lecture 1

Midterm I. October 13, 2010

This is a closed notes/ closed book/ electronics off exam. Please write a detailed proof/explanation for every problem. *Answers without an explanation will get very little partial credit.*

Please write legibly and cross out anything that you do not want the grader to read.

Notation: $\mathbb{R}$ — the set of real numbers, $\mathbb{Z}$ — the set of integer numbers; $\mathbb{R}_+$ — positive real numbers, $\mathbb{Z}_+$ — positive integer (i.e. natural) numbers.

You can use the quantifiers $\forall$ (meaning "for all" or "for any"), and $\exists$ (meaning "there exists" or "there is"), but please do not use $\not\forall$ or $\nexists$.

You can use all the usual properties of real and integer numbers: the properties of addition, multiplication, inequalities; that the sum (or product) of even numbers is even, etc.

Each problem is worth 10 points.

Name:

Problem	1	2	3	4	5	6	Total
Grade							

Problem 1. a) Write the negation of the following statement, first in plain English, and then using quantifiers and formulas, with no words.

For any element a of the set A and any element b of the set B, a is less than b.

- There exists an element a of the set A and an element b of the set B such that a is greater than or equal to b

- $\exists\, a \in A, \exists\, b \in B, a \geq b$

b) Write a statement logically equivalent to the one below, but using no negatives, first in plain English, and then using quantifiers and formulas, with no words.

There is no positive real number x such that for any natural number n, x is not greater than $1/n$.

If there is no number x such that P, this means that for all x, not P. Thus the equivalent statement is

- For all positive real numbers x there exist a natural number n such that x is greater than $1/n$

- $\forall\, x \in \mathbb{R}_+, \exists\, n \in \mathbb{Z}_+, x > \frac{1}{n}$

Problem 2. Let the sequence $\{a_k\}$ be defined by $a_1 = 2$, $a_{k+1} = 2a_k - 1$. Guess a general formula for a_k and prove it using induction.

We start by writing down the first few terms of the sequence:

$a_1 = 2 \quad a_2 = 3 \quad a_3 = 5 \quad a_4 = 9 \quad a_5 = 17$

Comparing this with $1, 2, 4, 8, 16, \ldots$ we guess the formula $a_k = 2^{k-1} + 1$.

We now prove this by induction:

1) For $k = 1$ $a_1 = 2$, and our formula gives $2^{1-1} + 1 = 1 + 1 = 2$, so OK

2) Assume the formula is true for k, so $a_k = 2^{k-1} + 1$. We then compute

$$a_{k+1} = 2a_k - 1 = 2 \cdot (2^{k-1} + 1) - 1$$
$$= 2 \cdot 2^{k-1} + 2 - 1 = 2^k + 1$$

which is exactly what the formula for $k+1$ should be.

The inductive proof is thus complete. $\square$

Problem 3. a) Prove that for any sets A, B, C, and D, if $C \subseteq A \cap B$ and $A \cup B \subseteq D$, then $A - B \subseteq D - C$.

You can use Venn diagrams to illustrate your proof, but Venn diagrams alone will not be considered a valid proof: you can use truth tables, algebra of sets, etc. Hint: think of D as the "universal" set.

b) Is $\big((A \cup B \subseteq D) \text{ and } (A - B \subseteq D - C)\big) \implies (C \subseteq A \cap B)$ true?

a) We consider D as the universal set, since it contains $A \cup B$, and thus A and B, and since $C \subseteq A \cap B$ is then also contained in D. We denote by $X^c = D - X$ the complement of X in D. Then we need to prove that if $C \subseteq A \cap B$, then $A - B \subseteq C^c$. Here is the truth table: notice that $C \subseteq A \cap B$ means that $x \in C \implies (x \in A)$ and $(x \in B)$, so we do not have all possible rows.

$x \in C$	$x \in A$	$x \in B$	$x \in A - B$	$x \in C^c$
T	T	T	F	F
F	?	?	?	T

In the second row we do not really care whether x is in $A - B$ — it is certainly in C^c, so indeed $A - B \subseteq C^c$.

b) This is false: take $A = B = \emptyset$ and $C = D \neq \emptyset$

Problem 4. For each of the following statements determine whether it is true or false, and justify your answer.

a) $\exists x \in \mathbb{Z}_+,\ \forall y \in \mathbb{Z}_+,\ x \leq y$ True False

Take $x = 1$; all natural numbers y are greater than or equal to 1.

b) $\exists x \in \mathbb{R}_+,\ \forall y \in \mathbb{R}_+,\ x \leq y$ True False

For any $x \in \mathbb{R}_+$ $y = \frac{x}{2}$ violates this statement.

c) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ xy \leq x$ True False

For any x we can take $y = 1$, so $xy = x \leq x$

d) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ xy \leq x$ True False

We can take $x = 0$, so that for any y we have $xy = 0 \leq x = 0$

Problem 5. For any pair of integers n and m prove that if n^2+n^3+m is even, then m is even. Let $A = n^2+n^3+m$ – it is even.

We consider first the case when n is odd. In that case n^2 is odd (as a product of two odd numbers), and n^3 is also odd. Thus n^2+n^3, being the sum of two odd numbers, is even. Then $m = A - (n^2+n^3)$ is even as the difference of two even numbers.

If n is even, then both n^2 and n^3 are even, and so their sum n^2+n^3 is also even, and we proceed as above.

Problem 6. Let $f : A \to B$ be a bijective function, and let $g : B \to A$ be a surjective function. For each of the two compositions $f \circ g$ and $g \circ f$, determine the domain and codomain, and whether the function is necessarily injective/surjective/bijective. Justify your answers by proofs or counterexamples.

The map $f \circ g : B \to B$ has B as both its domain and codomain.
The map $g \circ f : A \to A$ has A as both its domain and codomain.
Since both f and g are surjective (recall that f being bijective means it is both injective and surjective), their compositions are surjective. We discussed it in class, but here is a proof: we show $\forall a \in A \; \exists x \in A$, $g \circ f(x) = g(f(x)) = a$.
Indeed, since g is surjective, $\exists y \in B$ such that $g(y) = a$. But then since f is surjective, $\exists x \in A$ such that $f(x) = y$, and then $g(f(x)) = g(y) = a$.

Now let $A = B = \mathbb{Z}_+$, let f = identity map (so $f(x) = x$ for all x), and let

$$g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Since $g(2) = g(1)$, g is not injective.
But then $f \circ g = g \circ f = g$ are also not injective — and thus not bijective, either.