RESEARCH STATEMENT

RUSSELL LODGE

1. Introduction

My research interfaces between geometry, topology, group theory, and holomorphic dynamics. I am especially interested in the interaction between Teichmüller theory and holomorphic dynamics, as exemplified by W. Thurston's characterization and rigidity theorem for rational maps.

In the early 1980's, Douady and Hubbard demonstrated how finite trees can be used to model Julia sets of postcritically finite polynomials [8]. The resulting classification of postcritically finite polynomials in terms of "Hubbard trees" is a prototypical example of the connection between combinatorics and complex dynamics [2, 30]. Another bridge between the two disciplines is found in "spider theory", where binary sequences are used to encode postcritically finite polynomials with a single critical point, again by means of a finite graph. These classifications are highly significant, as it is conjectured that the set of maps that are quasiconformally conjugate to postcritically finite maps in a neighborhood of their Julia set are dense in parameter space [23, Conjecture 1.1]. However, in contrast to the rich combinatorial results for complex polynomials, far less is known about the dynamics of non-polynomial holomorphic functions.

To better understand dynamical properties of a postcritically finite rational function, it is often convenient to work with less rigid topological models. Such models can be produced, for example, by mating two postcritically finite polynomials [26], or by means of a finite subdivision rule of the two-sphere [5]. Thurston's theorem characterizes those postcritically finite branched covers that are equivalent to rational maps in terms of the mapping properties of multicurves. His theorem also provides a rigidity result—if the branched cover is equivalent to a rational function, this rational function is essentially unique (ignoring a family of well-understood Euclidean examples). This remarkable theorem is proven in [9], relying heavily on the dynamical properties of an analytic pullback map on the Teichmüller space associated to the sphere with marked set given by the postcritical points.

Thurston's pullback map is mysterious, and I seek a better understanding of its dynamics. The Weil-Petersson completion of Teichmüller space was shown by Selinger [34] to be the "right" boundary to consider in the context of Thurston's theorem, and there are strong reasons to carefully study the extension of the pullback map to this boundary. I also seek to better understand the notion of equivalence used in Thurston's theorem; great progress was made recently when the "twisted rabbit problem" was solved using the theory of self-similar groups, but it continues to be the case that little is known about non-polynomial maps. I plan to use these tools to classify and better understand specific families of rational maps, for example the families of nearly Euclidean Thurston maps and

Newton maps discussed below. Finally I intend to pursue combinatorial classification results in the setting of transcendental entire functions with escaping singular orbit, a study which will also shed light on the topology of parameter space. The pursuit of all these goals will illuminate and exploit rich connections between combinatorics, Teichmüller geometry, group theory, and arithmetic dynamics.

2. Thurston's Theorem

Denote the standard two-sphere by S^2 and the Riemann sphere by $\widehat{\mathbb{C}}$. Let $F : S^2 \to S^2$ be a finite degree branched covering, where C_F is the set of critical points. The *postcritical set* of F is given by

$$P_F = \bigcup_{i>0} F^{\circ i}(C_F).$$

The map F is called a *Thurston map* if the postcritical set is finite; we now define the equivalence relation used in Thurston's theorem.

Definition 1: Let F and G be Thurston maps with postcritical sets P_F and P_G respectively. Then F is *Thurston equivalent* to G if there are orientation preserving homeomorphisms $h_0, h_1 : (S^2, P_F) \longrightarrow (S^2, P_G)$ with h_0 homotopic to h_1 rel P_F, so that the following commutes:

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{h_0} & (S^2, P_G) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle G} \\ (S^2, P_F) & \xrightarrow{h_1} & (S^2, P_G). \end{array}$$

A curve in $S^2 \setminus P_F$ is called *essential* if it is closed, simple, and both components bounded by γ contain at least two postcritical points. A *multicurve* is a collection of disjoint, essential curves where no two curves in the collection are homotopic in $S^2 \setminus P_F$. An *obstructing multicurve* is an F-invariant multicurve that satisfies certain mapping properties not specified here.

Thurston's Theorem: *Let F be a Thurston map which is not a Lattès map. Then F is Thurston equivalent to a rational map if and only if there are no obstructing multicurves. Furthermore, if F is equivalent to a rational map, this rational map is unique up to Möbius conjugation.*

An important step in proving this theorem is to show that F is a rational map if and only if the associated pulback map Σ_F on Teichmüller space has a fixed point. Evidently the mapping properties of essential curves under preimage of F have deep implications for Σ_F, and we pursue this theme in both of the following sections. By way of further motivation, it should be noted that mapping properties of simple closed curves play an important role in the three other groundbreaking theorems proved by Thurston relating geometry and topology, as noted by Hubbard [15].

3. Boundary Values of the Pullback Map

Thurston's pullback map is complicated—in all known (nontrivial) cases it is infinite-to-one, and a number of recent works are devoted to its study [4, 34, 19, 20]. Selinger showed that the pullback map extends to the Weil-Petersson boundary, but few explicit computations of the boundary values were made for several years. Described below is an algorithm from [20] to compute the Weil-Petersson boundary values for any Thurston map with four postcritical points, assuming an algebraic object called the virtual endomorphism on moduli space is computed (in theory this can always be done). I have also contributed to the study of a special class of Thurston maps where the boundary values can be computed using completely different methods from the algorithm just mentioned [11]. A Thurston map is called *nearly Euclidean* if all critical points are simple and the postcritical set has exactly four points; an algorithm to compute the boundary values of the pullback map has been implemented [6, 27], heavily exploiting the close connection to Euclidean maps.

We briefly describe how to identify the Weil-Petersson boundary of Teichmüller space with the extended rational numbers $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\}$, for a Thurston map F with $|P_F| = 4$. Fix a (branched) double cover of S^2 by the torus $\mathbb{T}$ having branch values P_F. The slope of an essential curve γ is defined to be the slope of one of its lifts in $\mathbb{T}$ with respect to some choice of homology basis (using the standard methods of [10]). Both lifts of γ are mutually homotopic in $\mathbb{T}$, so slope in $S^2 \setminus P_F$ is well-defined. The Weil-Petersson metric is known to be incomplete, and for some identification of Teichmüller space with the upper half-plane, the Weil-Petersson boundary consists of the extended rational numbers $\overline{\mathbb{Q}}$ equipped with the horoball topology. We make the identification so that points lying in a "small" horoball tangent to $\frac{p}{q}$ correspond to complex structures on $S^2 \setminus P_F$ whose curve of slope $\frac{p}{q}$ has a "short" representative in its homotopy class.

Having identified the Weil-Petersson boundary with $\overline{\mathbb{Q}}$, we recall that Selinger showed Thurston's pullback map Σ_F extends to the Weil-Petersson boundary for any Thurston map F. A consequence of his work is that $\Sigma_F(\frac{p}{q}) = \frac{p'}{q'}$ if the curve of slope $\frac{p}{q}$ has an essential preimage under F of slope $\frac{p'}{q'}$ (if there is no essential preimage, then $\frac{p}{q}$ is mapped to the interior of Teichmüller space). Thus, an understanding of essential preimages of curves under F yields an understanding of the boundary values of the extended pullback map and vice versa. We define the *slope function* $\sigma_F : \overline{\mathbb{Q}} \cup \{O\} \to \overline{\mathbb{Q}} \cup \{O\}$ by declaring that $\sigma_F(\frac{p}{q}) = \frac{p'}{q'}$ if $\frac{p'}{q'}$ is the slope of the essential preimage of $\frac{p}{q}$; furthermore $\sigma_F(\frac{p}{q}) = O$ if there is no essential preimage (i.e. the preimages are all nullhomotopic, or bound a single postcritical point), and $\sigma_F(O) = O$. When F has more than four postcritical points, there is a natural generalization of the slope function called the *pullback relation on curves.*

An outline of the algorithm to compute $\sigma_f(\frac{p}{q})$ is sketched here. One first encodes $\frac{p}{q}$ as a group element g using the even continued fraction expansion of $\frac{p}{q}$. Applying the Reidemeister-Schreier algorithm, g can be written in terms of the generators of the domain of the virtual endomorphism φ_f. Finally, $\varphi_f(g)$ is computed, and a rational number is obtained by reversing the first step of the algorithm for the group element $\varphi_f(g)$; this rational number is shown in [20] to be precisely $\sigma_f(\frac{p}{q})$.

The virtual endomorphism methods have been implemented in a growing number of cases, including all quadratic rational maps with four postcritical points [17] and the noteworthy rational map $f(z) = \frac{3z^2}{2z^3+1}$ of [20]. This latter map f was earlier studied in [4], where among other things the Thurston pullback map Σ_f is depicted and shown to be surjective. I have proven that the slope function σ_f is surjective as well, has infinite fiber at every point, and has surprising global dynamics as described in the following theorem [20].

Theorem: *Let $\frac{p}{q} \in \overline{\mathbb{Q}}$ be a reduced fraction, and $f(z) = \frac{3z^2}{2z^3+1}$. Then under iteration of σ_f, $\frac{p}{q}$ maps into the two-cycle $\frac{0}{1} \leftrightarrow \frac{1}{0}$ or to the fixed point $\frac{1}{1}$. Specifically, $\frac{p}{q}$ maps to $\frac{1}{1}$ under iteration if and only if p and q are odd.*

Inspired by this example, it is said that a slope function σ_F has a *finite global attractor* if there is some finite set $\mathcal{A} \subset \overline{\mathbb{Q}} \cup \{O\}$ so that $\sigma_F(\mathcal{A}) \subset \mathcal{A}$, and some iterate of $\frac{p}{q}$ is contained in $\mathcal{A}$ for all $\frac{p}{q} \in \overline{\mathbb{Q}}$. Extensive computer experiments in the context of nearly Euclidean Thurston maps suggest that the slope function for a rational nearly Euclidean map has a finite global attractor [27].

4. The Twisting Problem

In addition to being interesting in their own right, slope functions provide a useful invariant for the equivalence used in Thurston's theorem. Thurston equivalence remains mysterious, and many natural problems related to it are quite difficult to solve. For example, the "Twisted Rabbit problem" is to determine whether the rabbit polynomial composed with an arbitrary Dehn twist is equivalent to the rabbit, corabbit, or airplane polynomial. Pilgrim described the lack of solution to the Twisted Rabbit Problem for over a decade a "humbling reminder" of the lack of suitable invariants [28]. The solution came in the work of Bartholdi and Nekrashevych, who introduced the permutational biset and the iterated monodromy group as invariants of Thurston equivalence for quadratic polynomials [1]. Their machinery effectively reduced the topological question of determining Thurston class to the algebraic question of determining nuclei of iterated monodromy groups. Though these methods work well in the setting of polynomials, it is unclear how to generalize them to the case of rational functions. I have contributed to solutions in several low degree cases [20, 17]

For example, the solution I gave to the twisting problem for $f(z) = \frac{3z^2}{2z^3+1}$ uses the dynamics of the slope function as an invariant for Thurston equivalence [20]. The pure mapping class group $\mathrm{PMCG}(\widehat{\mathbb{C}}, P_f)$ is defined to be the set of self-homeomorphisms of $\widehat{\mathbb{C}}$ that fix P_f pointwise, modulo isotopy fixing P_f. When $|P_f| = 4$, it is known that this group is isomorphic to the free group on two generators [10]; we fix two generators α and β of $\mathrm{PMCG}(\widehat{\mathbb{C}}, P_f)$. Following Bartholdi and Nekrashevych, the virtual endomorphism is extended to a function $\overline{\psi} : \mathrm{PMCG}(\widehat{\mathbb{C}}, P_f) \longrightarrow \mathrm{PMCG}(\widehat{\mathbb{C}}, P_f)$ in such a way that $g \circ f$ is Thurston equivalent to $\overline{\psi}(g) \circ f$. The following theorem from [20] describes the dynamics of $\overline{\psi}$.

Theorem: *For any $g \in PMCG(\widehat{\mathbb{C}}, P_f)$ there is a positive number N so that $\overline{\psi}^{\circ n}(g) \in \mathfrak{M}$ for all $n > N$ where*

$$\mathfrak{M} = \{e, \beta, \alpha^{-1}, \alpha^2\beta^{-1}, \alpha^{-1}\beta\alpha^{-1}, \alpha\beta^{-1}, \beta^2\} \cup \{\alpha(\beta\alpha)^k : k \in \mathbb{Z}\}$$

Thus the twisting problem reduces to identifying the Thurston class of each element of $\mathfrak{M}$ applied to f. It is shown that composing each element of $\{\alpha(\beta\alpha)^k : k \in \mathbb{Z}\}$ with f produces a one-parameter family of obstructed and pairwise inequivalent maps. If on the other hand $h \in \{e, \beta, \alpha^{-1}, \alpha^2\beta^{-1}, \alpha^{-1}\beta\alpha^{-1}, \alpha\beta^{-1}, \beta^2\}$, the Thurston class of $h \circ f$ is not immediately evident. It is possible to show that $h \circ f$ is unobstructed using wreath recursions and that it must be equivalent to f itself or to a second fixed rational function g. However, repeated attempts with known methods were unable to determine whether it was equivalent to f or g. The needed invariant was found, though, when the pullback on curves for g was shown to have two distinct two-cycles. Thus, if the pullback of $h \circ f$ on curves has two two-cycles, it cannot be equivalent to f because the finite global attractor for σ_f has only one two-cycle. This fact along with the fact that g is not a mating allows for a complete solution to the twisting problem for f, and it brings to light a valuable invariant of Thurston equivalence.

5. Newton Maps

A central motivation for Thurston's theorem is its usefulness in the combinatorial classification of families of rational maps; one particularly large and naturally motivated family is discussed here.

A rational function of degree greater than two is called a *Newton map* if it can be written in the form $z - \frac{p(z)}{p'(z)}$ where p is a polynomial. This Newton map is precisely the function used by Newton's method to find the roots of p. Each root of p is an attracting fixed point of the Newton map, and each immediate basin of such a fixed point contains the repelling fixed point ∞ in its boundary. All the finite fixed points of a postcritically finite Newton map are in fact superattracting, and the degree of the Newton map is equal to the degree of p. Tan Lei has given a classification of postcritically finite cubic Newton maps by establishing a bijection between the Newton maps and certain finite connected forward invariant graphs containing the postcritical set [35].

Moving to arbitrary degree, Mikulich, Rückert, and Schleicher give a classification of postcritically finite Newton maps with all critical points eventually *fixed* [25]; the essential piece of combinatorial data is the channel diagram. The vertices of this channel diagram are ∞ and the roots of p; the edges are given by arcs connecting the roots to ∞ through all possible accesses in the immediate basins. To capture the behavior of critical points that eventually land on the channel diagram, a sufficient number of preimages of the channel diagram are taken, and the connected component of this iterated preimage containing the channel diagram is called the Newton graph. Considered abstractly, this graph along with a graph map inherited from the dynamics of the Newton map has been shown to be a complete invariant for postcritically *fixed* Newton maps.

Most recently, I have taken part in the classification of all postcritically *finite* Newton maps of any degree [22, 21]. This is done by establishing a bijection between Newton maps up to affine conjugacy and certain finite forward-invariant graphs containing the postcritical set up to some Thurston-like equivalence. The graphs (equipped with self-maps) are called *extended Newton graphs*, and have three types of edges:

- *Newton graph edges* capturing the behavior of eventually fixed critical points,
- *Hubbard tree edges* arising from renormalizations at the free critical points, and
- *Newton ray edges* connecting the Hubbard trees to the Newton graph.

The admissible extended Newton graphs are classified explicitly in terms of combinatorics, and such graphs are called *abstract extended Newton graphs*. Thurston's theorem is used to show that an abstract graph gives rise to a rational map; in particular, Thurston obstructions are shown not to exist using a theorem about intersections of obstructions and invariant arc systems [29].

Theorem (Classification of postcritically finite Newton maps): *There is a natural bijection between postcritically finite Newton maps (up to Möbius conjugacy) and abstract extended Newton graphs (up to combinatorial equivalence) so that for every abstract extended Newton graph* (Σ, f)*, the associated postcritically finite Newton map has the property that its extended Newton graph is combinatorially equivalent to* (Σ, f)*.*

With a complete combinatorial invariant in hand, there are a number of natural and long-standing questions about Newton maps that should now be pursued (see Section 7)

6. Transcendental maps with escaping post-singular orbit

The discussion so far has been exclusively about rational functions; we now let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. By studying iteration on an infinite-dimensional Teichmüller space, I seek to classify (in terms of escape speed and combinatorics) a wide variety of entire functions whose singular orbits escape. Recall that a *singular value* of f is a point $p \in \mathbb{C}$ so that for every open neighborhood U of p, there is some component V of $f^{-1}(U)$ so that $f|_V$ is not bijective.

The *escaping set* of f is given by $I(f) := \{z : f^{\circ n}(z) \to \infty\}$. Similar to the polynomial case, it is known that the Julia set of f is equal to $\partial I(f)$, but the differences between polynomial and entire transcendental dynamics quickly emerge. The strong conjecture of Eremenko is the following: every point in $I(f)$ can be joined to ∞ by a curve in $I(f)$. It is widely known that the conjecture holds in the polynomial case, and the significance of "external rays" for polynomial rays cannot be over-emphasized. Nevertheless, the conjecture was shown to be false when f is transcendental entire, though it holds for a broad class of functions [32], namely all functions f of both finite type (the set of singular values is finite) and finite order ($|f(z)| < \exp(|z|^d)$ for z large in modulus).

Following [33], we construct dynamical rays for the exponential family $E_\lambda(z) = \lambda e^z, \lambda \in \mathbb{C} \setminus \{0\}$ with implications for the Eremenko conjecture. The construction can easily be generalized to functions of finite type and finite order. Note that E_λ has only one singular value. For fixed $z_0 \in \mathbb{C}$, let $z_n := E_\lambda^{\circ n}(z_0)$; then if the orbit $\{z_n\}$ escapes, it easily follows that $\{\text{Re} z_n\} \to +\infty$. Let H be the half plane $H := \{z| \text{ Re}(z) > \text{Re}(\lambda) + 1\}$; we call each component U_i of $E_\lambda^{-1}(H)$ a *tract*. Let $I_{\{i_k\}}$ be the set of esaping points that visit the tracts U_{i_k} in consecutive order. The points of $I_{\{i_k\}}$ can be ordered by rate of escape, and omitting the least element, the resulting ray terminates at ∞ and has topology compatible with the topology coming from order of speed of escape. This curve is called a *ray tail*, and any preimage of a ray tail contains a ray tail. The maximal extension of a ray tail under preimages is called a *dynamic ray*. For functions of finite type and finite order it has been shown that any escaping point is either on a dynamic ray or an endpoint of such a ray, establishing the Eremenko conjecture for this large class [32].

The sequence $\{i_k\}$ recording the tracts visited by an escaping value is called its *external address*. For the exponential family E_λ, a complete classification of all possible external addresses for escaping values is given in [33]. It is then natural to generalize this result to other parameter spaces (in the sense of Eremenko and Lyubich), where two entire functions f and g are in the same parameter space if there exist quasiconformal homeomorphisms $\phi, \psi : \mathbb{C} \to \mathbb{C}$ with $f = \phi^{-1} \circ g \circ \psi$. In particular, for any given map in a fixed parameter space, one should provide a combinatorial classification of external addresses for which rays have escaping endpoints. This classification is greatly aided by a powerful rigidity result of Rempe which asserts that the set of admissible external addresses for any given map is consistent throughout parameter space (he also shows that when restricting to a particular neighborhood of ∞, two maps within the same parameter space are quasiconformally conjugate) [31]. I am interested in classifying (in terms of external address and escape speed) all maps in a given parameter space whose singular values all escape. This will be carried out in collaboration with D. Schleicher using the following ideas.

We discuss the exponential family for simplicity, though the methods will undoubtedly apply to other parameter spaces (in fact a classification for exponential maps already exists [12, 14], but it uses ad-hoc means that don't generalize easily). First, a given admissible external address and rate of escape are used to produce an approximate singular orbit (as in [33]). If an exponential map realizes these combinatorics, there is a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ sending the approximate orbit to the actual orbit (i.e. there is a finite Teichmüller distance between the two marked complex planes). This determines which infinite dimensional Teichmüller space should be used (different rates of escape yield different Teichmüller spaces). Next, the pullback mapping σ is defined on this Teichmüller space in terms of the external address. We must show existence and uniqueness of a fixed point for σ; it corresponds to a unique member of parameter space realizing the combinatorics. To prove existence, σ is shown to be locally strictly contracting and a σ-invariant compact neighborhood is constructed about the approximate orbit. Uniqueness follows from the fact that two escaping orbits realizing a given external address and escape speed are related by a quasiconformal map that is asymptotically conformal; Strebel's frame mapping theorem then implies the existence of a Teichmüller geodesic connecting the approximate and actual orbits in Teichmüller space. The estimates used in constructing ϕ leave plenty of room to spare, strongly suggesting that the methods generalize to a much broader class of functions.

Ultimately, our goal is to apply Douady's principle of "plowing in dynamical plane to harvest in parameter space" to transcendental entire functions (see [13]). Fixing external addresses and all but one escape rate, the resulting one parameter family of entire maps should form a curve analogous to the stretching rays of Branner and Hubbard [3]. From here a study can be made of continuous dependence on escape rates and external address. There is every reason to believe that the structure of the escape locus will be rich, and the object of much fruitful study.

7. Future Research

- *Does the pullback relation on curves for any rational Thurston map that is not a Lattès map have a finite global attractor?* Beyond the rational map f discussed above, the global attractors for slope functions have been computed in a handful of examples, and they are all finite [28]; Pilgrim also proves that quadratic polynomials with periodic critical points have a finite global attractor. Extensive computer experiments on nearly Euclidean Thurston maps seem to indicate an affirmative answer to this question as well. A positive answer to this question would have implications for twisting problems (as demonstrated earlier) and questions related to polynomial mating. For example, it is known that a hyperbolic rational Thurston map F is a mating if and only if there is some invariant oriented curve under preimage which is called the equator [24]. Thus, if there are no fixed curves mapping by maximal degree in the finite global attractor, one is assured that F is not a mating.

There are a number of possible approaches to this question. The analytic properties of maps (or correspondences) on moduli space in [18] have implications, as do algebraic properties such as contraction of virtual endomorphisms on word length [1]. The former was used to show the existence of a finite global attractor [28], whereas the latter can compute the elements of this attractor explicitly. What makes f particularly interesting is that its virtual endomorphism is not contracting, and the known methods for understanding the analytic properties of the map on moduli space seemed to be inconclusive, yet the finite global attractor was able to be computed using some ad hoc algebra. This suggests that there might be a weaker contraction property for virtual endomorphisms that may be useful, or there may be some analytic properties that need to be discovered.

Another more topological approach to these questions is available when there is a well-understood forward invariant connected graph containing the postcritical set. Such graphs have been constructed in many cases [16, 7, 22]. These graphs can then be used in some cases to draw conclusions about the location of equators, and to prove the existence of a finite global attractor.

- *Can checking the hypothesis of Thurston's characterization theorem be reduced to understanding where the pre-images of a finite number of curves lie, and the degree by which they map?* In the setting of nearly Euclidean Thurston maps, [27] has computed such curves for hundreds of examples. Fixing the identification of Teichmüller space and the upper half plane as before, it is shown in [6] that a Euclidean horoball tangent to $\frac{p}{q}$ in Teichmüller space is mapped by Thurston's pullback map into a horoball of computable radius tangent to $\sigma_F(\frac{p}{q})$. This fact, together with some conformal modulus arguments, allow one to locate half-planes in Teichmüller space where no fixed points can exist for Σ_F and typically one can conclude that the fixed point lies in some hyperbolic polygon of finite hyperbolic diameter. I hope to investigate how well these methods can be adapted to other families of Thurston maps.
- *Which Newton maps are matings, and what can be said about the two polynomials in the mating?* A simple criterion already exists to determine whether a postcritically finite cubic Newton map is a mating: the only needed information is the elementary mapping properties of the unique non-fixed critical point [35]. For quartic maps however, the situation is already far more complicated, and it is expected that the answer will come in terms of the full combinatorial invariant from [21]. Recent attention has been given to the types of iterated monodromy groups that may

arise for Newton maps, and in the case where all critical points are periodic, it is likely that this groundbreaking tool will provide substantial aid. For example, a necessary condition for such a map to be a mating is the existence of an element in the iterated monodromy group that acts transitively on every level.

It has also been conjectured that two polynomials whose mating is a Newton must be of a certain type: namely, one of the polynomials must fix all of its critical points. This is certainly the case for cubic polynomials [35], and we are now able to investigate the conjecture in higher degree. It is also expected that many Newton maps arise as matings in more than one way, i.e. "shared matings". A characterization of such Newton maps in terms of their combinatorial invariant would provide a great wealth of examples to study this surprising and mysterious phenomenon.

- *The classification methods of Section 6 can certainly be adapted to parameter spaces beyond the exponential family—which parameter spaces?* Some natural explicit candidates are the cosine family and $z \mapsto p(z)e^z$ for some polynomial $p(z)$. The major difficulty here is to find a description of the asymptotics of escaping singular orbits that is good enough to specify the orbits up to quasiconformal equivalence in $\mathbb{C}$; the Teichmüller iteration theory should be analogous to the exponential family. Beyond these and other explicit families, I want to investigate more broadly whether the results can be extended to maps of finite type and finite order (the maps that were so successfully studied in [32] and to what extent these classification techniques can account for singular values that escape on endpoints of rays.

References

[1] L. Bartholdi and V. Nekrashevych. Thurston equivalence of topological polynomials. *Acta Math.*, (197):1–51, 2006.

[2] B. Bielefeld, Y. Fisher, and J. Hubbard. The classification of critically preperiodic polynomials as dynamical systems. *J. Amer. Math. Soc.*, 5:721–762, 1992.

[3] B. Branner and J. Hubbard. The iteration of cubic polynomials, part i: The global topology of parameter space. *Acta Mathematica*, 160(1):143–206, 1988.

[4] X. Buff, A. Epstein, S. Koch, and K. Pilgrim. On Thurston's pullback map. In D. Schleicher, editor, *Complex dynamics: families and friends*, pages 561–583. A K Peters, Wellesley, MA, 2009.

[5] J. W. Cannon, W. J. Floyd, and W. R. Parry. Finite subdivision rules. *Conform. Geom. Dyn.*, 5:153–196, 2001.

[6] J.W. Cannon, W.J. Floyd, K. Pilgrim, and W.R. Parry. Nearly Euclidean Thurston maps. *Conform. Geom. Dyn.*, 16:209–255, 2012.

[7] K. Cordwell, S. Gilbertson, N. Nuechterlein, K. Pilgrim, and S. Pinella. On the classification of critically fixed rational maps. *Preprint*, 2013. ArXiv:1308.5895.

[8] A. Douady and J. Hubbard. Étude dynamique des polynômes complex I & II. *Publ. Math. Orsay*, 1984,1985.

[9] A. Douady and J. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.*, (171):263–297, 1993.

[10] B. Farb and D. Margalit. *A Primer on Mapping Class Groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, 2011.

[11] W.J. Floyd, G. Kelsey, S. Koch, R. Lodge, W. Parry, K. Pilgrim, and S. Saenz-Maldonado. Origami, affine maps, and complex dynamics. *in preparation.*

[12] M. Förster. *Exponential maps with escaping singular orbits.* PhD thesis, 2006. International University Bremen.

[13] M. Förster, L. Rempe, and D. Schleicher. Classification of escaping exponential maps. *Proc Amer Mathe Soc.*, (136):651–663, 2008.

[14] M. Förster and D. Schleicher. Parameter rays in the space of exponential maps. *Ergod Theory Dynam Systems*, (29):515–544, 2009.
[15] J. Hubbard. *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*. Matrix Editions, 2006.
[16] J. Hubbard and D. Schleicher. The spider algorithm. In *Complex dynamical systems*, pages 155–180. Amer. Math. Soc., 1994.
[17] G. Kelsey and R. Lodge. Quadratic nearly Euclidean Thurston maps. *In preparation*, 2016.
[18] S. Koch. Teichmüller theory and critically finite endomorphisms. *Advances in Mathematics*, 248:573–617, 2013.
[19] S. Koch, K. Pilgrim, and N. Selinger. Pullback invariants of Thurston maps. *Transactions of the American Mathematical Society*, To appear.
[20] R. Lodge. Boundary values of the Thurston pullback map. *Conform. Geom. Dyn.*, 17:77–118, 2013.
[21] R. Lodge, Y. Mikulich, and D. Schleicher. A classification of postcritically finite Newton maps. *Preprint*. arXiv:1510.02771.
[22] R. Lodge, Y. Mikulich, and D. Schleicher. Combinatorial properties of Newton maps. *Preprint*. arXiv:1510.02761.
[23] C. McMullen. Complex dynamics and renormalization. *Annals of Math Studies*, 135, 1994.
[24] D. Meyer. Unmating of rational maps, sufficient criteria and examples. In A. Bonifant, M. Lyubich, and S. Sutherland, editors, *Frontiers in Complex Dynamics: In Celebration of John Milnors 80th Birthday*, pages 197–234. Princeton University Press, 2014.
[25] Y. Mikulich, J. Rückert, and D. Schleicher. A combinatorial classification of postcritically fixed Newton maps. *Provisionally accepted*.
[26] J. Milnor. Pasting together Julia sets: A worked out example of mating. *Experimental Mathematics*, (13:1):55, 2000.
[27] W. Parry and W Floyd. Netmap. www.math.vt.edu/netmaps/, 2016.
[28] K. Pilgrim. An algebraic formulation of Thurston's characterization of rational functions. *Annales de la Faculte des Sciences de Toulouse*, 6XXI(5):1033–1068, 2012.
[29] K. Pilgrim and L. Tan. Combining rational maps and controlling obstructions. *Ergodic Theory Dynam. Systems*, (18):221–245, 1998.
[30] A. Poirier. On postcritically finite polynomials, part 2: Hubbard trees. *Stony Brook IMS preprint 93/7*, 1993.
[31] L. Rempe. Rigidity of escaping dynamics for transcendental entire functions. *Acta Mathematica*, (203):235–267, 2009.
[32] G. Rottenfüsser, J. Rückert, L. Rempe, and D. Schleicher. Dynamic rays of bounded-type entire functions. *Annals of Mathematics*, (173):77–125, 2011.
[33] D. Schleicher and J. Zimmer. Escaping points for exponential maps. *J. Lond. Math. Soc.*, 2(67):380–400, 2003.
[34] N. Selinger. Thurston's pullback map on the augmented Teichmüller space and applications. *Inventiones mathematicae*, pages 1–32, 2011.
[35] L. Tan. Branched coverings and cubic Newton maps. *Fundamenta mathematicae*, 154:207–259, 1997.