

QUADRATIC THURSTON MAPS WITH FEW POSTCRITICAL POINTS

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ABSTRACT. We use the theory of self-similar groups to enumerate all combinatorial classes of non-exceptional quadratic Thurston maps with fewer than five postcritical points. The enumeration relies on our computation that the corresponding maps on moduli space can be realized by quadratic rational maps with fewer than four postcritical points.

Holomorphic dynamics in one complex variable is largely concerned with the study of rational maps as dynamical systems, along with their parameter spaces. Postcritically finite rational maps have attracted much attention due to their relative simplicity and their structural significance in parameter space. Furthermore, W. Thurston has given a powerful characterization and rigidity theorem for postcritically finite rational maps up to combinatorial equivalence (see Theorem 1.1). With this tool at our disposal, we are free to consider rational maps in the more flexible topological category.

Definition (Thurston map). Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be an orientation-preserving branched cover of finite degree $d \geq 2$ whose set of critical points is denoted C_f . Then the *postcritical set* of f is given by $P_f := \bigcup_{i>0} f^{oi}(C_f)$. If P_f is finite, f is said to be a *Thurston map*.

Thurston maps will be considered up to the following equivalence relation, which can roughly be thought of as “conjugacy up to homotopy.”

Definition (Combinatorial equivalence). Two Thurston maps f and g are said to be *combinatorially equivalent* if and only if there exist homeomorphisms $h_0, h_1 : (\mathbb{S}^2, P_f) \rightarrow (\mathbb{S}^2, P_g)$ so that

$$h_0 \circ f = g \circ h_1$$

and h_0 is isotopic to h_1 through a one parameter family of homeomorphisms $h_t : (\mathbb{S}^2, P_f) \rightarrow (\mathbb{S}^2, P_g)$ where $0 \leq t \leq 1$.

It is natural to seek an enumeration of all combinatorial classes (see for instance Problem 5.2 posed by McMullen [14]). Few explicit listings exist, with the following exceptions. First is the enumeration of all non-polynomial hyperbolic rational maps of degree 3 or less with four or fewer postcritical points [4]. This is done by listing all possible ramification portraits (i.e. combinatorial descriptions of critical and postcritical dynamics), and then finding coefficients of the corresponding rational functions by solving equations derived from the portraits. However, enumeration is substantially more difficult in the presence of combinatorial classes with no rational representative. The only existing results of this type come in the context of quadratic Thurston maps with four postcritical points [8, 3]. For example, Bartholdi and Nekrashevych enumerate all maps with the same ramification portrait as $z^2 + i$ using special features of self-similar groups associated to polynomials.

We enumerate the equivalence classes of all non-exceptional quadratic Thurston maps with four or fewer postcritical points (see Theorem 1.2). Thurston maps with three or fewer postcritical points are all equivalent to rational maps, so in this case our enumeration amounts to listing all possible ramification portraits and computing coefficients. For non-exceptional maps with four postcritical points, we rely on a crucial fact about the transitivity of the mapping class action. A *twist* of f is a map of the form $h \circ f$ where $h : (\mathbb{S}^2, P_f) \rightarrow (\mathbb{S}^2, P_f)$ is a homeomorphism with $h|_{P_f} = id$; note that $h \circ f$ is again a Thurston map with the same ramification portrait. A special feature of quadratic Thurston maps with four postcritical points is that every combinatorial class having the same ramification portrait as f can be represented by a twist of f (see [7, §5]).

We present an algorithm to determine the combinatorial class of some twist h_0 of f . Recall that the pure mapping class group of (\mathbb{S}^2, P_f) is isomorphic to the free group on two generators. Following [3], we extend the virtual endomorphism on the pure mapping class group and iterate. This defines a sequence of mapping classes $\{h_i\}_{i=0}^\infty$ where $h_i \circ f$ is combinatorially equivalent to $h_{i+1} \circ f$ for all $i \geq 0$. The nucleus of the self-similar action of the pure mapping class group on combinatorial classes contains the tails of all such sequences $\{h_i\}_{i=0}^\infty$. We prove that the mapping class biset is *sub-hyperbolic*, which means that the nucleus is either finite or has only finitely-many distinct actions of the mapping class group. We compute the nucleus for representative quadratic Thurston maps, and show that for any Thurston map f it consists of a finite number of cycles of the extended virtual endomorphism together with at most two \mathbb{Z} -parameter families of fixed points. With a few exceptions, each of these cycles correspond to distinct combinatorial classes, and we identify rational representatives (if they exist) using biset invariants such as the pullback function on curves discussed below.

The mapping class bisets we use in our calculations are also the central object in Bartholdi and Dudko's work on the decidability of combinatorial equivalence [2]. These bisets are interesting objects in their own right. The question of which Thurston maps give rise to sub-hyperbolic mapping class bisets was asked in [17], and our work provides the first answer for non-polynomial maps.

The correspondence on moduli space [9] plays a central role in our enumeration of combinatorial classes, much as it did in [3]. As a special feature, each correspondence is actually a restriction of a quadratic map with at most three postcritical points. We show that quadratic Thurston maps with four postcritical points exist under a hierarchy of successively finer invariants:

- number of critical postcritical points
- conjugacy class of map on moduli space
- ramification portrait.

Furthermore, every quadratic rational map with four postcritical points is written as the composition of a Möbius transformation and a quadratic rational map with fewer postcritical points (see e.g. Proposition 2.1). Using this additional structure, we efficiently choose connecting paths so that all wreath recursions take a particularly simple form.

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1. THURSTON THEORY

A Thurston map is called *Euclidean* if its orbifold Euler characteristic is zero (cf. [5, Definition 1.1]). A quadratic Thurston map with four postcritical points is Euclidean if and only if it has no periodic critical point.

We will state Thurston's theorem in the special case when f is a non-Euclidean Thurston map with $|P_f| = 4$ (the general statement is found in [6]). A simple closed curve in \mathbb{S}^2 is said to be *essential* if it separates the postcritical set into two pairs of points. Two essential curves are said to be *homotopic* if they are homotopic in $\mathbb{S}^2 \setminus P_f$. The collection of all homotopy classes of essential curves is denoted \mathcal{C}_f .

Definition. The *pullback function on curves* $\mu_f : \mathcal{C}_f \cup \{\odot\} \rightarrow \mathcal{C}_f \cup \{\odot\}$ is defined as follows:

- if $\gamma \in \mathcal{C}_f$, and $f^{-1}(\gamma)$ has an essential component $\tilde{\gamma}$, then $\mu_f(\gamma) = \tilde{\gamma}$
- if $\gamma \in \mathcal{C}_f$, and $f^{-1}(\gamma)$ does not have an essential component, then $\mu_f(\gamma) = \odot$
- $\mu_f(\odot) = \odot$.

Let $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ be the collection of essential components of $f^{-1}(\gamma)$, and let d_i denote the mapping degree of $\tilde{\gamma}_i \rightarrow \gamma$. Then $\gamma \in \mathcal{C}_f$ is said to be an *obstructing curve for f* if $\mu_f(\gamma) = \gamma$ and $\lambda_f := \sum_{i=1}^k \frac{1}{d_i} \geq 1$.

Theorem 1.1 (Thurston's theorem). *Let f be a Thurston map with four postcritical points that is not Euclidean. Then f is equivalent to a rational map if and only if there is no obstructing curve. If it exists, the rational map is unique up to Möbius conjugacy.*

rational maps	ramification portrait	fixed points
$(1 - 2z)^2$	$\frac{1}{2} \implies 0 \longrightarrow 1 \curvearrowright \infty \curvearrowright$	$\frac{1}{4}, 1, \infty$
$\frac{1}{1 - (1 - 2z)^2}$	$\frac{1}{2} \implies 1 \longrightarrow \infty \iff 0$	$\approx -.4196, .7098 \pm .3031i$
$1 - \frac{1}{(1 - 2z)^2}$	$\frac{1}{2} \implies \infty \longrightarrow 1 \longrightarrow 0 \curvearrowright$	$0, 1 \pm \frac{1}{2}i$
z^2	$0 \curvearrowright \infty \curvearrowright$	$0, 1, \infty$
$1 - z^2$	$0 \iff 1 \quad \infty \curvearrowright$	$\frac{1}{2}(-1 \pm \sqrt{5}), \infty$
$\frac{1}{z^2}$	$0 \iff \infty$	$1, \frac{1}{2}(-1 \pm \sqrt{3}i)$
$\frac{1}{1 - z^2}$	$\infty \iff 0 \iff 1$	$\approx -1.347, 0.6624 \pm .5623i$

TABLE 1. All rational maps in $Q(2) \cup Q(3)$ up to automorphism. The first three maps have one critical postcritical point, and the last four maps have two.

Applications of this theorem include the combinatorial classifications of large families of postcritically finite rational maps [22, 13].

Denote by $Q(i)$ the set of quadratic Thurston maps with exactly i postcritical points. Clearly $Q(1)$ is empty. Every element of $Q(2) \cup Q(3)$ is combinatorially equivalent to a rational map, and an exhaustive list of rational representatives is given in Table 1. Rational Euclidean maps in $Q(4)$ are Lattès maps, and have been classified up to conjugacy [15]. However, there are infinitely many combinatorial classes of Euclidean Thurston maps not equivalent to rational maps. Since $Q(4)$ maps are NET maps and the study of combinatorial classes of such maps can be reduced to the study of similarity classes of matrices, the question of enumeration of classes becomes a question of the number of distinct eigenvalues for a given trace [19]. Thus, the classification amounts to solving a class number problem, a notoriously difficult type of problem in number theory that we prefer to avoid [11, 20]. Thus our main focus will be the set of non-Euclidean quadratic Thurston maps, denoted $Q(4)^*$.

Theorem 1.2 (Quadratic enumeration). *Let $f \in Q(2) \cup Q(3) \cup Q(4)^*$. If f is not obstructed, it is equivalent to exactly one rational map in Table 1, 2, or 3. If f is obstructed, it is equivalent to exactly one obstructed map in Table 5.*

Ramification portraits. We now describe an invariant for Thurston maps that is central to the organization of our study.

Let f be a Thurston map. *The ramification portrait* of f is a directed labeled graph which we denote G_f . The vertices of G_f are given by the points in $Z_f := P_f \cup C_f$. A directed edge connects vertex v to vertex w if and only if $f(v) = w$. The label of such an edge is given by the local degree of f at v .

We say that Thurston maps f and g have equivalent ramification portraits if and only if there is a bijection $h : Z_f \rightarrow Z_g$ so that $h \circ f = g \circ h$ on Z_f and h preserves edge labels (i.e. the local degree of f at $z \in Z_f$ is identical to that of g at $h(z)$). Clearly if Thurston maps f and g are combinatorially equivalent, they have equivalent ramification portraits.

2. RAMIFICATION PORTRAITS AND MAPS ON MODULI SPACE IN $Q(4)^*$

There are 13 possible ramification portraits for maps in $Q(4)^*$. Three portraits are realized by polynomials, and though the corresponding Thurston classes are enumerated in [3], we include them for completeness. We generate all ramification portraits in $Q(4)^*$ by composing a Möbius transformation and a rational map in $Q(2) \cup Q(3)$. This approach yields a natural organization of the portraits in group theoretic terms.

g -map	Möb	$Q(4)^*$ ramification portrait	rational maps
$(1-2z)^2$	$M_{\bullet,0}$	$\frac{1}{2} \implies \bullet \longrightarrow 0 \longrightarrow \infty \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 1$	$\bullet = \frac{1}{4}$
	$M_{\bullet,1}$	$\frac{1}{2} \implies \infty \implies 0 \longrightarrow \bullet \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 1$	$\bullet = \frac{1}{4}$
	$M_{\bullet,\infty}$	$\frac{1}{2} \implies 1 \longrightarrow 0 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \infty \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \bullet$	$\bullet = \frac{1}{4}$
$\frac{1}{1-(1-2z)^2}$	$M_{\bullet,1}$	$\frac{1}{2} \implies \bullet \longrightarrow 1 \longrightarrow 0 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \infty \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array}$	$\bullet \approx -.4196, .7098 \pm .3031i$
	$M_{\bullet,\infty}$	$\frac{1}{2} \implies 0 \longrightarrow \bullet \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \infty \implies 1$	$\bullet \approx -.4196, .7098 \pm .3031i$
	$M_{\bullet,0}$	$\frac{1}{2} \implies \infty \implies \bullet \longrightarrow 0 \longrightarrow 1 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array}$	$\bullet \approx -.4196, .7098 \pm .3031i$
$1 - \frac{1}{(1-2z)^2}$	$M_{\bullet,\infty}$	$\frac{1}{2} \implies \bullet \longrightarrow \infty \implies 0 \longrightarrow 1 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array}$	$\bullet = 1 \pm \frac{1}{2}i$
	$M_{\bullet,0}$	$\frac{1}{2} \implies 1 \longrightarrow \bullet \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 0 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \infty \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array}$	$\bullet = 1 \pm \frac{1}{2}i$
	$M_{\bullet,1}$	$\frac{1}{2} \implies 0 \longrightarrow \infty \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \bullet \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 1$	$\bullet = 1 \pm \frac{1}{2}i$

TABLE 2. All rational maps $f \in Q(4)^*$ with $|P_f \cap C_f| = 1$ presented as the composition of a map $g \in Q(3)$ and a Möbius transformation which depends on some choice of a point $\bullet \in \mathbb{C}$ fixed by g (see Proposition 2.1). More generally, the portrait of any Thurston map in $Q(4)^*$ with one critical postcritical point appears in this table. Double arrows indicate nontrivial ramification.

For each ramification portrait, we compute the correspondence on moduli space in the sense of [9] (see also [7, §1]). This correspondence is crucial for enumerating the obstructed $Q(4)^*$ maps in §5, as well as computing the pullback on curves in §6 (see e.g. [3, 21, 12]). For any member of $Q(4)^*$, the correspondence on moduli space restricts to a quadratic rational map. It is common for maps in $Q(4)^*$ with different ramification portraits to have conjugate maps on moduli space. We normalize maps on moduli space so that conjugate maps are identical. In this way, we organize $Q(4)^*$ combinatorial classes under the following successively finer invariants:

- number of critical postcritical points (i.e. impure Hurwitz class, cf. [7, Theorem 7]),
- map on moduli space,
- ramification portrait (i.e. pure Hurwitz class, cf [7, §5]).

Let $M_{\bullet,0}$, $M_{\bullet,1}$, and $M_{\bullet,\infty}$ respectively denote the Möbius transformations inducing the following permutations, where $\bullet \notin \{0, 1, \infty\}$:

$$(\bullet 0)(1 \infty), \quad (\bullet 1)(0 \infty), \quad (\bullet \infty)(0 1).$$

One critical postcritical point. Each ramification portrait in Table 2 (i.e. the ramification of each map $f \in Q(4)^*$ with $|P_f \cap C_f| = 1$) arises as the ramification portrait of some map $M_{\bullet,i} \circ g$ where

- $g \in Q(3)$ so that $|P_g \cap C_g| = 1$ (i.e. the first three maps in Table 1),
- $\bullet \notin P_g$ is a point satisfying $g(\bullet) = \bullet$ (see third column of Table 1),
- $i \in \{0, 1, \infty\}$.

Specifically, one finds g in the first column, and $M_{\bullet,i}$ in the second column.

Two critical postcritical points. Each ramification portrait in Table 3 (i.e. the ramification of each map $f \in Q(4)^*$ with $|P_f \cap C_f| = 2$) arises as the ramification portrait of some map $M_{\bullet,i} \circ g$ where

- $g \in Q(2) \cup Q(3)$ so that $|P_g \cap C_g| = 2$ (i.e. the last four maps in Table 1),
- $\bullet \notin \{0, 1, \infty\}$ is a point satisfying $g(\bullet) = \bullet$, and
- $i \in \{0, 1, \infty\}$.

Note that we must proceed formally for the map z^2 since no fixed point \bullet exists.

g -map	Möb	$Q(4)^*$ ramification portrait	rational maps
z^2	$M_{\bullet, \infty}$	$0 \rightleftarrows 1 \quad \infty \rightleftarrows \bullet$	N/A
$1 - z^2$	$M_{\bullet, 1}$	$\infty \rightleftarrows 0 \implies \bullet \implies 1$	$\bullet = \frac{1}{2}(-1 \pm \sqrt{5})$
$\frac{1}{z^2}$	$M_{\bullet, 0}$	$0 \rightleftarrows 1 \implies \infty \implies \bullet$	$\bullet = \frac{1}{2}(-1 \pm \sqrt{3}i)$
$\frac{1}{1-z^2}$	$M_{\bullet, 1}$	$0 \rightleftarrows \bullet \implies 1 \quad \infty \curvearrowright$	$\bullet \approx -1.347, 0.6624 \pm .5623i$

TABLE 3. All rational maps $f \in Q(4)^*$ with $|P_f \cap C_f| = 2$ as the composition of a map $g \in Q(2) \cup Q(3)$ and a Möbius transformation which depends on a point $\bullet \in \mathbb{C}$ fixed by g . The portrait of any Thurston map in $Q(4)^*$ with two critical postcritical points appears in this table (no rational map has the first ramification portrait).

Group action. The vertex labels of ramification portraits in Tables 2 and 3 were assigned in a particular way. One motivation is that the ramification portraits admit a group action as described here. When $|P_f \cap C_f| = 1$, the Möbius transformation inducing the permutation $(0 \ 1 \ \infty)$ acts by postcomposition on the set of ramification portraits, where the portrait in row i is sent to the portrait in row $i + 3 \pmod 9$. Furthermore, when $|P_f \cap C_f| = 2$, the Möbius transformation inducing the transposition $(0 \ \infty)$ acts by postcomposition on ramification portraits where row i is sent to row $i + 2 \pmod 4$.

Maps on moduli space. Another motivation for our particular choice of vertex labels comes from the fact that we can succinctly describe normalizations for the maps on moduli space so that Möbius conjugate maps are in fact equal.

The notation g in the first part of this section was used suggestively, i.e. in appropriate coordinates g is the map on moduli space for the corresponding $f \in Q(4)^*$. Suppose the ramification portrait for f is given in terms of a composition of the Möbius transformation $M_{\bullet, i}$ and some quadratic branched cover g with three or fewer postcritical points as before. Let $\tau_g : \mathbb{S}^2 \rightarrow \widehat{\mathbb{C}}$ be a point in the Teichmüller space of the sphere \mathbb{S}^2 marked by $\{0, 1, \infty, \bullet\}$, and let $\tilde{\tau}_g := \sigma_g(\tau_g)$ denote the image of τ_g under the pullback map on Teichmüller space (see [9] for all relevant definitions). Normalize as follows:

$$\tau_g(0, 1, \infty, \bullet) = (0, 1, \infty, y)$$

$$\tilde{\tau}_g(0, 1, \infty, \bullet) = (0, 1, \infty, x).$$

Then normalize the markings τ_f and $\tilde{\tau}_f := \sigma_f(\tau_f)$ as follows:

$$\tau_f = \tau_g \circ M_{\bullet, i}^{-1}$$

$$\tilde{\tau}_f = \tilde{\tau}_g.$$

One can check directly that the map on moduli space for f is given identically by g .

Example: Let $g(z) = \frac{1}{z^2}$ and let $f = M_{\bullet, 0} \circ g$ where $\bullet \notin \{0, 1, \infty\}$ is some fixed point of g . Subject to the normalizations above, the rational map $\tau_f \circ f \circ \tilde{\tau}_f^{-1}$ has the property that $(0, 1, \infty, x)$ is mapped to $(\infty, 1, 0, y)$, where 0 and ∞ are simple critical points. From this we see that $\tau_f \circ f \circ \tilde{\tau}_f^{-1}(z) = \frac{1}{z^2}$ and thus the map on moduli space is given by $y = \frac{1}{x^2}$ defined on the domain $\widehat{\mathbb{C}} \setminus \{0, \pm 1, \infty\}$.

Rational $Q(4)^*$ maps as compositions. Tables 2 and 3 indicate how all $Q(4)^*$ ramification portraits arise as the composition of a Möbius transformation and a quadratic rational map. Restricting to the case when a map in $Q(4)^*$ is rational, we strengthen the result to give a holomorphic decomposition of the rational map itself.

Proposition 2.1. *Let $f \in Q(4)^*$ be a rational map with $|P_f \cap C_f| = 1$. Then f is Möbius conjugate to a unique map of the form $M_{\bullet, i} \circ g$ where:*

- $g \in Q(3)$ is a rational map with $|P_g \cap C_g| = 1$,
- $\bullet \notin P_g$ is a fixed point of g , and

– $i \in \{0, 1, \infty\}$.

Proof. There is a unique portrait in Table 2 equivalent to the portrait of f . Each portrait is asymmetric, so without loss of generality assume f has been (uniquely) normalized to have vertex labels 0, 1, and ∞ as in the table portrait. Let \bullet be the fourth postcritical point of f uniquely determined by the normalization, and let $i := f(\bullet)$. Then for $j \in \{0, 1, \infty\}$, it is checked directly that $g := M_{\bullet, j} \circ f \in Q(3)$ if and only if $j = i$. Evidently g fixes \bullet , and $M_{\bullet, i} \circ g = f$. All the data in the proposition statement has thus been uniquely determined. \square

A similar statement holds for the two critical postcritical points with a similar proof, except that the uniqueness no longer holds due to portrait symmetry. For example, if $g(z) = 1 - z^2$ and \bullet is some fixed point, then $M_{\bullet, 1} \circ g$ and $M_{\bullet, 0} \circ g$ are Möbius conjugate.

3. BACKGROUND ON SELF-SIMILAR GROUPS

A self-similar group is understood through its action by isomorphisms on a tree. Let $X = \{1, 2, \dots, d\}$ be a finite alphabet. We identify the set X^* of finite words in X with the vertices of an infinite, $d + 1$ -regular, rooted tree. We will think of the tree as “growing” up from the root, so the children of each vertex will be “above” it. Note that an isomorphism of this tree will preserve the “levels” of the tree (i.e. the length of words in X^*). Further, the isomorphism will take the subtree above a vertex to the subtree above the image of that vertex (i.e. words with a given suffix will be taken to words that end with the image of that suffix). Since the subtree above any vertex of an infinite, $d + 1$ regular, rooted tree is canonically isomorphic to the full tree, we can identify the subtrees above the pre-image and image vertices with the full tree. Thus, the *restriction* of the isomorphism to the subtree above a particular vertex can be viewed as an element of the isomorphism group of the full tree.

A group action by isomorphisms on an infinite, $d + 1$ regular, rooted tree is *self-similar* if it is closed under restrictions. That is, the restriction of any element of the group to any vertex will be an element of the group.

An element of a self-similar group (i.e. a group admitting a self-similar action) can be described by giving the action of the element on the first level of the tree and the restrictions of the element to all vertices on that first level. This notation is called the *wreath recursion* of the element. For example, for an element h of a group with a self-similar action on a binary rooted tree, we would write

$$h = \langle h|_1, h|_2 \rangle \pi_h$$

where $h|_1$ represents the restriction of h to the vertex identified with 1, $h|_2$ the restriction of h to the vertex 2, and π_h is the permutation on $\{1, 2\}$ induced by h . We omit π_h if it is trivial, and we denote the non-trivial permutation on $\{1, 2\}$ by σ .

To multiply wreath recursions, use the following rule:

$$\langle g|_1, g|_2, \dots, g|_d \rangle \pi \cdot \langle h|_1, h|_2, \dots, h|_d \rangle \tau = \langle g|_1 h|_{\pi(1)}, g|_2 h|_{\pi(2)}, \dots, g|_d h|_{\pi(d)} \rangle \pi \tau$$

See section 1.4 of [16] for a discussion of the connection between self-similar groups and wreath products.

A *virtual endomorphism* of a group G is a homomorphism from a finite-index subgroup of G to G . Given a self-similar group G and a letter x of the tree with vertex set X^* that it acts on, the restriction map at x is a natural virtual endomorphism from the stabilizer of x to G . So, for example, suppose G acts on an infinite, binary, rooted tree whose vertices we identify with $\{1, 2\}^*$. Then any element of G can be written as $h = \langle h|_1, h|_2 \rangle \pi_h$. The virtual endomorphism associated with the first coordinate has domain equal the set of elements $h \in G$ such that π_h is trivial and such an element is mapped to $h|_1$.

Given a virtual endomorphism ϕ of a group G , one can construct a self-similar action of G on a tree with vertex set X^* , where X is the set of cosets of the domain of ϕ . For details see section 2.5 of [16].

A self-similar action of G on a tree X^* is said to be *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $h \in G$ there exists an $n_h \in \mathbb{N}$ such that $h|_v \in \mathcal{N}$ for all words $v \in X^*$ with length at least n_h . The minimal such set \mathcal{N} is unique and is called the *nucleus* of the self-similar action. The self-similar action is contracting if and only if the *contracting coefficient*

$$\rho = \limsup_{n \rightarrow \infty} \sqrt[n]{\limsup_{l(h) \rightarrow \infty} \max_{v \in X^n} \frac{l(h|_v)}{l(h)}}$$

is strictly less than 1. One should think of a contracting self-similar action as one where the restrictions of a group element get simpler (i.e. shorter in word length) as you restrict further up the tree. At some height of the tree, all of these restrictions will be in a finite set (the nucleus).

It is important to note that the nucleus of a contracting self-similar action is defined to be a finite set. In this work, we will be making use of a set with the same definition as the nucleus but we will not require the set to be finite. We will refer to this set as the *(possibly infinite) nucleus*.

We will briefly describe another formulation of the self-similar theory. Given a self-similar action of a group G on a tree X^* , we can define a G -biset (sometimes called a *bimodule*) as follows: the set itself is $G \times X$, with elements written as formal products $h \cdot x$. There is a left action of G by left multiplication, and a right action of G by

$$(g \cdot x) \cdot h = gh|_x \cdot h(x)$$

If the associated self-similar action is contracting, we say that the biset is *hyperbolic*. If the action is not contracting, but the action by the faithful quotient is, then we say that the biset is *sub-hyperbolic*. In other words, in the sub-hyperbolic case the infinite nucleus contains only finitely-many distinct actions. All of the bisets for the actions we discuss in this paper are either hyperbolic (when the nucleus is finite) or sub-hyperbolic (when the nucleus is infinite). See Table 4 and Theorem 4.1 for details.

In [3], the authors use this self-similar machinery to solve the twisting problem for all polynomials in $Q(4)$. Specifically, they take a virtual endomorphism ϕ of the pure mapping class group G and extend it to a map (not a homomorphism) $\bar{\phi} : G \rightarrow G$ such that for $T \in G$, $T \circ f$ is equivalent to $\bar{\phi}(T) \circ f$. Thus, the question of equivalence class of the twists can be reduced to those twists that are periodic under $\bar{\phi}$. Since the G -biset of the self-similar action associated with ϕ is sub-hyperbolic in these cases, this set is relatively straightforward to analyze.

4. WREATH RECURSIONS FOR MAPS ON MODULI SPACE

We first represent pure mapping classes of (\mathbb{S}^2, P_f) as elements of the fundamental group of moduli space. This induces an isomorphism between the mapping class biset of f and the fundamental group biset of the map on moduli space at an appropriate basepoint according to [3] and [12, Theorem 2.6]. Wreath recursions for all maps on moduli space at all fixed points are computed, and their nuclei are computed in a generalized sense. These nuclei will be vital for both our solution to the twisting problem and our computation of the global dynamics of the pullback on curves.

Twisting in terms of fundamental group. The point-pushing isomorphism is used here to write the left action of the pure mapping class group as a right action of the fundamental group of moduli space. Let \mathcal{M}_f denote moduli space (identified with $\mathbb{C} \setminus \{0, 1\}$ as before) and suppose z_0 is the fixed point of the map on moduli space g corresponding to some map $f \in Q(4)^*$. Let $\gamma : [0, 1] \rightarrow \mathcal{M}_f$ be a loop with $\gamma(0) = \gamma(1) = z_0$. Then γ extends to a motion $\tilde{\gamma} : \mathcal{M}_f \times [0, 1] \rightarrow \mathcal{M}_f$ that “pushes” z_0 along γ ; in other words $\tilde{\gamma}(\cdot, t)$ is a homeomorphism and $\tilde{\gamma}(z_0, t) = \gamma(t)$ for all t . Then to $[\gamma] \in \pi_1(\mathcal{M}_f, z_0)$ we associate the isotopy class of $T_\gamma := \tilde{\gamma}(\cdot, 1)$. This is a pure mapping class of (\mathbb{S}^2, P_f) under the identification of $\widehat{\mathbb{C}}$ with \mathbb{S}^2 via the stereographic projection. We use the notation $f \cdot \gamma := T_\gamma \circ f$, following [12, §8] and [3].

Generators and connecting paths. We need to compute a wreath recursion for each map on moduli space based at each fixed point outside of $\{0, 1, \infty\}$ (suitably interpreted for $g(z) = z^2$). Such wreath recursions depend on non-canonical choices of generators and connecting paths; in the interest of streamlining our presentation we fix generators $\hat{\alpha}, \hat{\beta}$ in Figure 1 and then use the Möbius action to produce generators for all other maps on moduli space (labelled α and β). Connecting paths will be chosen so as to simplify the wreath recursion presentations, and in particular the first coordinate of every active element will be trivial (i.e. $\alpha|_1 = 1$). This fact will greatly simplify the computations in the proofs of Lemmas 5.2, 6.1, and 6.2. All wreath recursions and their nuclei are recorded in Table 4.

One critical postcritical point. The map $g_1(z) = (1-2z)^2$ has exactly one non-postcritical fixed point, namely $1/4$. The three maps on moduli space that arise in Table 2 have the form $M \circ g_1$ where M is the unique Möbius transformation inducing the permutations $id, (0\ 1\ \infty)$ and $(0\ \infty\ 1)$ respectively.

Now let z_0 be some fixed point of $M \circ g_1$ outside $\{0, 1, \infty\}$. If z_0 can be connected to $M(1/4)$ by a line segment in \mathbb{C} that doesn’t pass through 0 or 1, let ℓ be this line segment (clearly such line segments exist

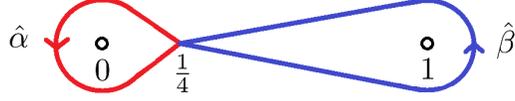


FIGURE 1. Generators $\hat{\alpha}$ and $\hat{\beta}$ of $\pi_1(\mathbb{C} \setminus \{0, 1\}, 1/4)$



FIGURE 2. Lifts of generators from Figure 1 under g_1

when $z_0 \notin \mathbb{R}$). Otherwise, fix a path in $\mathbb{R} \cup \{\infty\}$ connecting z_0 to $M(1/4)$ that passes through exactly one point in $\{0, 1, \infty\}$ which we denote p ; then ℓ is taken to be this path perturbed to the left of p . In either case, we fix $\alpha := \ell \cdot M(\hat{\alpha}) \cdot \bar{\ell}$ and $\beta := \ell \cdot M(\hat{\beta}) \cdot \bar{\ell}$ as the generators of $\pi_1(\mathbb{C} \setminus \{0, 1\}, z_0)$, where $\bar{\ell}$ denotes the reverse of ℓ .

For the wreath recursion computation, the label 1 is assigned to the fixed point z_0 and the label 2 is assigned to the other preimage of z_0 under $M \circ g_1$. The connecting paths for $M \circ g_1$ are taken to be the constant path at z_0 and the unique lift of α under $M \circ g_1$ based at z_0 .

Two critical postcritical points. The map $g_2(z) = z^2$ does not have any fixed points outside of $\{0, 1, \infty\}$. We construct the connecting paths for g_2 explicitly since it is exceptional. Let ℓ_0 be a path in \mathbb{R} connecting $1/4$ to $1/2$ and let ℓ_1 be the path connecting $1/4$ to $-1/2$ along a semi-circular arc in the upper half-plane.

Note that the three other maps on moduli space in Table 3 have the form $M \circ g_2$ where M is the Möbius transformation inducing the permutations $(0\ 1)$, $(0\ \infty)$, and $(0\ 1\ \infty)$ respectively. The construction of wreath recursions is carried out as in the case of one critical postcritical point.



FIGURE 3. Lifts of generators from Figure 1 under g_2

Nuclei. Having determined the wreath recursions, we compute each (possibly infinite) nucleus using a technique described in Lemma 2.11.2 in [16]. Specifically, we create a candidate nucleus N by closing a generating set under the operations of restriction and inversion (such a set is *state-closed*). Usually we use $\{1, \alpha, \beta\}$ for this set, but sometimes β will not be in the nucleus. In this case, we use $\{1, \alpha, \gamma\}$ where $\gamma = \beta^{-1}\alpha^{-1}$ or $\{1, \alpha, \delta\}$ where $\delta = \alpha^{-1}\beta^{-1}$.

We then check every element of N^2 to determine if there is a number n such that after *any* n restrictions the result is in N . If this property holds for all elements of N^2 , then N contains our nucleus, since we can use N as a generating set and produce contraction in word length. If this property does not hold, there will be some cycles of elements of N^2 under restrictions. We create a new candidate nucleus by including these cycles in N and we repeat the process.

For example, consider the wreath recursion $\alpha = \langle 1, \gamma \rangle \sigma$, $\beta = \langle 1, \beta \rangle$, which arises from the g -map $\frac{1}{z^2}$ with fixed point $\frac{1}{2}(-1 - \sqrt{3}i)$. Set $N = \{1, \alpha, \beta^n, \gamma \mid n \in \mathbb{Z}\}^{\pm 1}$, since α restricts to γ and all powers of β will self-restrict. When checking products in the set N^2 , we see that some of them do not restrict back into N . Since N is state-closed, all of the restrictions of N^2 do fall in N^2 , however. The restrictions of N^2 that do *not* fall in N are displayed below (we do not display the symmetric relations on the inverses).

$$\alpha\gamma^{-1} \rightarrow \gamma\beta \rightarrow \gamma^{-1}\beta \quad \gamma^{-2} \rightarrow \gamma\beta \quad \beta^n\alpha \rightarrow \beta^n\gamma \quad \gamma\beta^n \rightarrow \alpha\beta^{n+1}$$

g -map	Fixed point	Wreath recursion	(Possibly infinite) “Nucleus”
$(1 - 2z)^2$	$\frac{1}{4}$	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \alpha, \beta \rangle$	$\{1, \alpha^n, \beta^n \mid n \in \mathbb{Z}\}$
$\frac{1}{1 - (1 - 2z)^2}$	-.4196	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \delta, \beta\alpha\beta^{-1} \rangle$	$\{1, \alpha, \beta, \delta, \beta\alpha\beta^{-1}, \delta^2, \alpha^{-1}\beta^{-2}, \beta\delta\beta^{-1}, \beta\alpha^2\beta\alpha, \delta^{-1}\beta\delta, \delta^{-1}\alpha\}$
	.7098 + .3031 <i>i</i>	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \alpha, \gamma \rangle$	$\{1, \alpha, \gamma\}^{\pm 1}$
	.7098 - .3031 <i>i</i>	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \alpha, \delta \rangle$	$\{1, \alpha, \delta\}^{\pm 1}$
$1 - \frac{1}{(1 - 2z)^2}$	$1 + \frac{i}{2}$	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \gamma, \beta \rangle$	$\{1, \alpha^n, \beta^n, \gamma^n, \beta^{-1}\alpha^n\beta, \alpha^n\beta, \alpha\beta^2 \mid n \in \mathbb{Z}\}^{\pm 1}$
	$1 - \frac{i}{2}$	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \delta, \beta \rangle$	$\{1, \alpha^n, \beta^n, \delta^n, \beta\alpha^n\beta^{-1}, \beta\alpha^n, \beta^2\alpha \mid n \in \mathbb{Z}\}^{\pm 1}$
z^2	N/A	$\alpha = \langle 1, \alpha \rangle \sigma, \quad \beta = \langle \beta, 1 \rangle$	$\{1, \alpha, \beta^n, \alpha\beta^n, \alpha\beta^n\alpha^{-1} \mid n \in \mathbb{Z}\}^{\pm 1}$
$1 - z^2$	$\frac{1}{2}(-1 + \sqrt{5})$	$\alpha = \langle 1, \beta \rangle \sigma, \quad \beta = \langle \alpha, 1 \rangle$	$\{1, \alpha, \beta, \alpha\beta^{-1}\}^{\pm 1}$
	$\frac{1}{2}(-1 - \sqrt{5})$	$\alpha = \langle 1, \beta \rangle \sigma, \quad \beta = \langle 1, \beta\alpha\beta^{-1} \rangle$	$\{1, \alpha, \beta, \delta, \beta\alpha\beta^{-1}\}^{\pm 1}$
$\frac{1}{z^2}$	$\frac{1}{2}(-1 + \sqrt{3}i)$	$\alpha = \langle 1, \delta \rangle \sigma, \quad \beta = \langle 1, \alpha^{-1}\beta\alpha \rangle$	$\{1, \alpha, \beta^n, \alpha^{-1}\beta^n\alpha, \beta^n\alpha \mid n \in \mathbb{Z}\}^{\pm 1}$
	$\frac{1}{2}(-1 - \sqrt{3}i)$	$\alpha = \langle 1, \gamma \rangle \sigma, \quad \beta = \langle 1, \beta \rangle$	$\{1, \alpha, \beta^n, \gamma \mid n \in \mathbb{Z}\}^{\pm 1}$
$\frac{1}{1 - z^2}$	-1.347	$\alpha = \langle 1, \delta \rangle \sigma, \quad \beta = \langle 1, \alpha \rangle$	$\{1, \alpha, \beta, \delta\}^{\pm 1}$
	0.6624 + .5623 <i>i</i>	$\alpha = \langle 1, \gamma \rangle \sigma, \quad \beta = \langle \alpha, 1 \rangle$	$\{1, \alpha, \beta, \gamma\}^{\pm 1}$
	0.6624 - .5623 <i>i</i>	$\alpha = \langle 1, \delta \rangle \sigma, \quad \beta = \langle \alpha, 1 \rangle$	$\{1, \alpha, \beta, \delta\}^{\pm 1}$

TABLE 4. Wreath recursions for the maps on moduli space at all points outside $\{0, 1, \infty\}$, and the (possibly infinite) “nucleus” for those recursions. Recall that $\gamma = \beta^{-1}\alpha^{-1}$ and $\delta = \alpha^{-1}\beta^{-1}$.

Since the longest path along these restrictions has length two, we can see that every element of N^2 will restrict into N after three restrictions. Thus N is our (infinite) nucleus.

We present the results of these computations in Table 4. Again, we point out that we are allowing our “nucleus” to be infinite, unlike in the standard literature.

Theorem 4.1. *The image of each of the “nuclei” (even the infinite ones) listed in Table 4 is finite in the faithful quotient of the self-similar action. That is, the bisets of these self-similar actions are all sub-hyperbolic.*

Proof. The faithful quotient of the action is the iterated monodromy group of the map g . Recall that g is always a member of $Q(2) \cup Q(3)$, so it is a postcritically finite rational function. Then by Theorem 6.4.4 of [16], the image of the nucleus in $IMG(g)$ is finite.

However, this result can also be seen directly from Table 4. Each element whose entire cyclic subgroup appears in a nucleus in Table 4 has order either 1, 2, or 4 in the faithful quotient. This can be checked using the wreath recursions given in the table (for example, any map with one critical postcritical point has α^2 acting trivially). \square

5. TWISTING $Q(4)^*$ MAPS

Recall that the twisting problem is the determination of the combinatorial class of a map $h \circ f$ where f is a Thurston map and h is a pure mapping class of (\mathbb{S}^2, P_f) . As above, we will use the identification $f \cdot \gamma := T_\gamma \circ f$ and treat h as an element of the fundamental group of moduli space.

We give an algorithm to solve this problem for all maps $f \in Q(4)^*$ in the manner of [3]: define a virtual endomorphism ϕ on $\pi_1(\mathcal{M}_f, z_0)$ (which we identify with the pure mapping class group) and extend it to a non-homomorphic map $\bar{\phi}$ on the entire group such that $f \cdot h \simeq f \cdot \bar{\phi}(h)$ where \simeq denotes combinatorial equivalence (we will use $=$ to denote homotopy equivalence). Then we find the attractor of forward iteration of $\bar{\phi}$ on the group and determine the equivalence class of each of the those twists on f .

The extended virtual endomorphism $\bar{\phi}$. For a map $f \in Q(4)^*$, we define the virtual endomorphism ϕ on $\pi_1(\mathcal{M}_f, z_0)$ as the first coordinate virtual endomorphism from the wreath recursion in Table 4. So for the map in $Q(4)^*$ defined by fixed point $1/4$ of the g -map $(1 - 2z^2)$ for example, the virtual endomorphism ϕ would have domain generated by $\alpha^2, \beta, \alpha^{-1}\beta\alpha$ and would map the generators by

$$(1) \quad \phi(\alpha^2) = 1, \phi(\beta) = \alpha, \phi(\beta\alpha) = \beta.$$

We then define a map (not a homomorphism) $\bar{\phi}$ from the pure mapping class group to itself:

$$\bar{\phi}(h) = \begin{cases} \phi(h) & \text{if } h \in \text{Dom}(\phi) \\ \alpha\phi(h\alpha^{-1}) & \text{if } h \in \text{Dom}(\phi)\alpha \end{cases}$$

Lemma 5.1. For $h \in \pi_1(\mathcal{M}_f, z_0)$,

$$f \cdot h \simeq f \cdot \bar{\phi}(h)$$

Proof. Notice if $h \in \text{Dom}(\phi)$, then h lifts under f to $\phi(h)$:

$$f \cdot h = \phi(h) \cdot f \simeq f \cdot \phi(h) = f \cdot \bar{\phi}(h)$$

and if $h \in \text{Dom}(\phi)\alpha$, then

$$f \cdot h = f \cdot h\alpha^{-1} \cdot \alpha = \phi(h\alpha^{-1}) \cdot f \cdot \alpha \simeq f \cdot \alpha\phi(h\alpha^{-1}) = f \cdot \bar{\phi}(h)$$

Thus, we have defined $\bar{\phi}$ such that $f \cdot h \simeq f \cdot \bar{\phi}(h)$ for all $h \in \pi_1(\mathcal{M}_f, z_0)$. \square

To solve the twisting problem, we will study the dynamics of $\bar{\phi}$ on $\pi_1(\mathcal{M}_f, z_0)$.

Lemma 5.2. Let $h \in \pi_1(\mathcal{M}_f, z_0)$. There exists N such that for all $n > N$

$$\bar{\phi}^{on}(h) \in \mathcal{N} \cup \alpha\mathcal{N}$$

Proof. Recall that we chose to make the wreath recursion of α of the form $\alpha = \langle 1, \alpha|_2 \rangle \sigma$. Notice that if $h = \langle h_1, h_2 \rangle$, then $\bar{\phi}(h) = h_1$ and

$$\bar{\phi}(\alpha h) = \alpha\phi(\alpha h\alpha^{-1}) = \alpha\phi(\langle 1, \alpha|_2 \rangle \sigma \langle h_1, h_2 \rangle \langle \alpha|_2^{-1}, 1 \rangle \sigma) = \alpha h_2$$

and if $h = \langle h_1, h_2 \rangle \sigma$, then

$$\bar{\phi}(h) = \alpha\phi(h\alpha^{-1}) = \alpha\phi(\langle h_1, h_2 \rangle \sigma \langle \alpha|_2^{-1}, 1 \rangle \sigma) = \alpha h_1$$

and

$$\bar{\phi}(\alpha h) = \phi(\alpha h) = \phi(\langle 1, \alpha|_2 \rangle \sigma \langle h_1, h_2 \rangle \sigma) = h_2$$

Thus, $\bar{\phi}(h)$ is either equal to a restriction of h or α times a restriction of h . Since taking repeated restrictions of h eventually enters (and remains in) \mathcal{N} , by the computations above we have that iterating $\bar{\phi}$ on any element of the fundamental group will eventually land in the set $\mathcal{N} \cup \alpha\mathcal{N}$. \square

Consequently, to solve the twisting problem we compute the action of $\bar{\phi}$ on the set $\mathcal{N} \cup \alpha\mathcal{N}$. For example, consider the virtual endomorphism defined by equations 1 above. The resulting dynamics of $\bar{\phi}$ on the set $\mathcal{N} \cup \alpha\mathcal{N}$ are:

$$\beta^{2n} \rightarrow \alpha^{2n} \rightarrow 1 \rightarrow 1 \quad \beta^{2n+1} \rightarrow \alpha^{2n+1} \rightarrow \alpha \rightarrow \alpha \quad \alpha\beta^n \rightarrow \alpha\beta^n$$

The periodic elements of this action are displayed in Table 6 for one representative of each pure Hurwitz class.

Remark 5.3. While we have only performed the $\bar{\phi}$ computations for one $Q(4)^*$ map within each pure Hurwitz class, we can write an element of any other combinatorial class as a twist of the first $Q(4)^*$ map, and then perform the same method. For instance, we made no $\bar{\phi}$ computations for the rational map f_- defined by the fixed point $\frac{1}{2}(-1 - \sqrt{5})$ of the g -map $1 - z^2$. However, for the rational map f_+ defined by the other fixed point of that g -map, we have that $f_- \simeq f_+ \cdot \alpha$ since α by definition has non-trivial monodromy action and there are no other $Q(4)^*$ maps in the pure Hurwitz class. To twist f_- by h , we simply use $\bar{\phi}^{on}(\alpha h)$ with $\bar{\phi}$ computed from f_+ and determine the resulting $\bar{\phi}$ cycle as usual.

Identifications. All that remains is to identify each $\bar{\phi}$ cycle with an equivalence class of Thurston maps. The solution to the twisting problem, then, is to apply $\bar{\phi}$ repeatedly to the specified twist until obtaining one of the periodic elements, each of which is identified with a combinatorial class.

Proof Theorem 1.2. Recall that any map in $Q(2)$ or $Q(3)$ is unobstructed, so from the list of ramification portraits in Table 1 one simply computes coefficients to give an enumeration. Each rational map in $Q(4)^*$ can be found the same way (see Table 2 and 3), and a cross ratio argument shows that any two such maps are not Möbius conjugate. From the preceding discussion, all obstructed maps in $Q(4)^*$ must lie in a $\bar{\phi}$ cycle, and the remainder of our proof is occupied with analyzing the corresponding Thurston classes (organized by map on moduli space). For all $\bar{\phi}$ attractors, the trivial element is identified with the combinatorial class of the $Q(4)^*$ map used in the $\bar{\phi}$ computations.

- $g(z) = (1 - 2z)^2$: the periodic element α is identified with the equivalence class of a Thurston map with obstruction β . Twisting about this obstruction produces an infinite family of distinct combinatorial classes of obstructed maps (see the argument in [10, Theorem 9.2 V]).
- $g(z) = \frac{1}{1-(1-2z)^2}$: we can distinguish between the combinatorial classes identified with α and with the 2-cycle by considering the finite global attractor (FGA) results proved in the next section. If f is the $Q(4)$ map defined by the fixed point with positive imaginary part (i.e. the $Q(4)$ map used to compute the virtual endomorphism ϕ from the first coordinate of the wreath recursion), then we can compute the FGA for $f \cdot \alpha$ by changing the virtual endomorphism to instead be associated with the second coordinate. When we make these calculations, we see that the FGA for $f \cdot \alpha$ has the 2-cycle $\beta \leftrightarrow \gamma$, so α is identified with the combinatorial class of the $Q(4)$ map defined by fixed point $-.4196$. Thus, the combinatorial class of the $Q(4)$ map defined by the fixed point with negative imaginary part is identified with the 2-cycle $\gamma^{-1} \leftrightarrow \alpha\gamma$.
- $g(z) = 1 - \frac{1}{(1-2z)^2}$: the 2-cycle is identified with the combinatorial class of the $Q(4)$ map defined by other fixed point of the g -map, and the $\alpha\beta^n$ are identified with an infinite family of combinatorial classes of obstructed maps as above.
- $g(z) = z^2$: in the row of Table 6 for the g -map z^2 , we have two families of fixed points of $\bar{\phi}$: β^n and $\alpha^2\beta^n\alpha^{-1}$ for $n \in \mathbb{Z}$. In fact, these are both identified with the same family of combinatorial classes of obstructed maps. Let f be the (obstructed) $Q(4)$ map used to make the $\bar{\phi}$ calculations and observe that since $\alpha^2 = \langle \alpha, \alpha \rangle$ for this wreath recursion, then

$$f \cdot \alpha^2\beta^n\alpha^{-1} = \alpha \cdot f \cdot \beta^n\alpha^{-1}$$

and by conjugation we have that

$$\alpha \cdot f \cdot \beta^n\alpha^{-1} \simeq f \cdot \beta^n$$

Thus, $f \cdot \beta^n$ and $f \cdot \alpha^2\beta^n\alpha^{-1}$ are equivalent maps.

- $g(z) = 1 - z^2$: let f be the $Q(4)$ map defined by the fixed point $\frac{1}{2}(-1 + \sqrt{5})$ of g . We can show that $f \cdot \alpha^2\beta^{-1} \simeq f$ (since $\alpha^2 = \langle \beta, \beta \rangle$ in this setting), and so the 2-cycle is identified with the combinatorial class of f . Thus, α is identified with the combinatorial class of the $Q(4)$ map defined by the other fixed point.
- $g(z) = \frac{1}{z^2}$: the 2-cycle is identified with the combinatorial class of the fixed point of the g -map with positive imaginary part, and the $\alpha\beta^n$ are identified with an infinite family of combinatorial classes of obstructed maps.
- $g(z) = \frac{1}{1-z^2}$: we again use the virtual endomorphism for the second coordinate to compute the FGA for the map identified with α . This computation reveals it to match the $Q(4)$ map defined by

g -map	Möb	$Q(4)^*$ ramification portrait	fixed point	obstructed twists
$(1 - 2z)^2$	$M_{\bullet,0}$	$\frac{1}{2} \implies \bullet \longrightarrow 0 \longrightarrow \infty \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} 1$	$\bullet = \frac{1}{4}$	$\alpha\beta^n$
	$M_{\bullet,1}$	$\frac{1}{2} \implies \infty \implies 0 \longrightarrow \bullet \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} 1$	$\bullet = \frac{1}{4}$	$\alpha\beta^n$
	$M_{\bullet,\infty}$	$\frac{1}{2} \implies 1 \longrightarrow 0 \curvearrowright \infty \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \bullet$	$\bullet = \frac{1}{4}$	$\alpha\beta^n$
$1 - \frac{1}{(1 - 2z)^2}$	$M_{\bullet,\infty}$	$\frac{1}{2} \implies \bullet \longrightarrow \infty \implies 0 \longrightarrow 1 \curvearrowright$	$\bullet = 1 + \frac{1}{2}i$	$\alpha\beta^n$
	$M_{\bullet,0}$	$\frac{1}{2} \implies 1 \longrightarrow \bullet \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} 0 \quad \infty \curvearrowright$	$\bullet = 1 + \frac{1}{2}i$	$\alpha\beta^n$
	$M_{\bullet,1}$	$\frac{1}{2} \implies 0 \longrightarrow \infty \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \bullet \longrightarrow 1$	$\bullet = 1 + \frac{1}{2}i$	$\alpha\beta^n$
z^2	$M_{\bullet,\infty}$	$0 \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} 1 \quad \infty \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \bullet$	See caption	β^n
$\frac{1}{z^2}$	$M_{\bullet,0}$	$0 \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} 1 \longrightarrow \infty \implies \bullet$	$\bullet = \frac{1}{2}(-1 - \sqrt{3}i)$	$\alpha\beta^n$

TABLE 5. One representative for each obstructed combinatorial class in $Q(4)^*$ can be found in this table. Representatives are expressed in terms of compositions, e.g. if $g(z) = (1 - 2z)^2$ then $T_{\alpha\beta^n} \circ M_{\bullet,i} \circ g$ is obstructed for all $n \in \mathbb{Z}$ and $i \in \{0, 1, \infty\}$ where $\bullet = \frac{1}{4}$ and T_γ is the twist along the closed curve γ (see beginning of Section 4 for details). Recall that the fixed point is taken to be the basepoint of the fundamental group with exception of z^2 where it is taken to be $\frac{1}{4}$ as before. Thus the definition of the symbols α and β vary as in §4).

the fixed point with negative imaginary part, and so the $Q(4)$ map defined by the fixed point with positive imaginary part is identified with the 3-cycle.

We have thus shown that each $\bar{\phi}$ cycle either corresponds to one of the rational maps already found, or lies in a one-parameter family of obstructed twists found in Table 5. \square

6. PULLBACK ON ESSENTIAL CURVES

Recall the definition of the pullback function on curves $\mu_f : \mathcal{C}_f \cup \{\odot\} \rightarrow \mathcal{C}_f \cup \{\odot\}$ from §1

Definition. A *finite global attractor* of the pullback function on curves μ_f is a finite, forward invariant set $\mathcal{A} \subset \mathcal{C}_f \cup \{\odot\}$ so that for each γ , there exists n so that $\mu_f^{\circ n}(\gamma) \in \mathcal{A}$.

Not every Thurston map has a finite global attractor for its pullback on curves (one example is given below). Pilgrim has conjectured that if f is a rational map with hyperbolic orbifold, then μ_f has a finite global attractor. The program NETmap [18] has provided a great deal of evidence in favor of this conjecture, and a number of results have been proven in special cases [21, 12, 10, 7].

We investigate the global dynamics of the pullback on curves for rational $Q(4)^*$ maps. All pullbacks have finite global attractors, and minimal ones are exhibited in Table 5. Our results verify Theorem 8 of [7], where it is shown that the minimal finite global attractor of any rational $Q(4)^*$ map with one critical postcritical point has at most four curves in \mathcal{C}_f , using entirely different methods from our own.

Computing the pullback on curves. Let $f \in Q(4)^*$ be a rational map. We will compute its pullback on curves in terms of the virtual endomorphism on moduli space using the naturality statement of [12, Theorem 2.6]. Let C be an essential curve in (\mathbb{S}^2, P_f) and denote by G the group $\pi_1(\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}, z_0)$; recall this is a free group generated by α and β . The point-pushing isomorphism (see section 4) identifies the right hand Dehn twist about C with a unique parabolic element of G having the form $g = w^{-1}xw$ where $x \in \{\alpha, \beta, \gamma\}$ and $w \in G$. In case C has an essential preimage, the pullback $\mu_f(C)$ is given by the essential curve corresponding to the parabolic element $\phi((x^n)^w)$, where n is minimal so that the expression is defined (i.e. $w^{-1}x^n w$ is in the domain of ϕ). If C has no essential preimage, then $\phi((x^n)^w)$ is trivial. The computation of the pullback

g -map	Wreath recursion	Attractor of the $\bar{\phi}$ map	FGA
$(1-2z)^2$	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \alpha, \beta \rangle$	$1 \curvearrowright, \alpha\beta^n \curvearrowright$ for $n \in \mathbb{Z}$	\odot
$\frac{1}{1-(1-2z)^2}$	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \delta, \beta\alpha\beta^{-1} \rangle$	————	$\odot, \beta \leftrightarrow \gamma$
	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \alpha, \gamma \rangle$	$1 \curvearrowright, \alpha \curvearrowright, \gamma^{-1} \leftrightarrow \alpha\gamma$	\odot
	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \alpha, \delta \rangle$	————	\odot
$1 - \frac{1}{(1-2z)^2}$	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \gamma, \beta \rangle$	$1 \curvearrowright, \gamma \leftrightarrow \alpha\gamma^{-1}, \alpha\beta^n \curvearrowright$ for $n \in \mathbb{Z}$	\odot
	$\alpha = \langle 1, 1 \rangle \sigma, \quad \beta = \langle \delta, \beta \rangle$	————	\odot
z^2	$\alpha = \langle 1, \alpha \rangle \sigma, \quad \beta = \langle \beta, 1 \rangle$	$\beta^n \curvearrowright, \alpha^2\beta^n\alpha^{-1} \curvearrowright$ for $n \in \mathbb{Z}$	N/A
$1 - z^2$	$\alpha = \langle 1, \beta \rangle \sigma, \quad \beta = \langle \alpha, 1 \rangle$	$1 \curvearrowright, \alpha \curvearrowright, \beta\alpha^{-1} \leftrightarrow \alpha^2\beta^{-1}$	$\odot, \alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta$
	$\alpha = \langle 1, \beta \rangle \sigma, \quad \beta = \langle 1, \beta\alpha\beta^{-1} \rangle$	————	$\odot, \delta \curvearrowright$
$\frac{1}{z^2}$	$\alpha = \langle 1, \delta \rangle \sigma, \quad \beta = \langle 1, \alpha^{-1}\beta\alpha \rangle$	————	$\odot, \alpha \leftrightarrow \delta$
	$\alpha = \langle 1, \gamma \rangle \sigma, \quad \beta = \langle 1, \beta \rangle$	$1 \curvearrowright, \gamma \leftrightarrow \alpha\gamma^{-1}, \alpha\beta^n \curvearrowright$ for $n \in \mathbb{Z}$	$\odot, \alpha \leftrightarrow \gamma$
$\frac{1}{1-z^2}$	$\alpha = \langle 1, \delta \rangle \sigma, \quad \beta = \langle 1, \alpha \rangle$	$1 \curvearrowright, \alpha \curvearrowright, \delta \leftrightarrow \gamma^{-1} \rightarrow \alpha^2$	\odot
	$\alpha = \langle 1, \gamma \rangle \sigma, \quad \beta = \langle \alpha, 1 \rangle$	————	$\odot, \alpha \leftrightarrow \gamma \rightarrow \beta$
	$\alpha = \langle 1, \delta \rangle \sigma, \quad \beta = \langle \alpha, 1 \rangle$	————	$\odot, \alpha \leftrightarrow \delta \rightarrow \beta$

TABLE 6. Wreath recursion for each map on moduli space at each fixed point not in $\{0, 1, \infty\}$, the attractor for the non-homomorphism map that is used to solve the twisting problem, and the minimal finite global attractor for the pullback map on curves. Note that we compute the $\bar{\phi}$ attractor for only one map in each pure Hurwitz class. This is all that is required for the enumeration; the other attractors can be computed using Remark 5.3 if desired. Recall that $\gamma = \beta^{-1}\alpha^{-1}$ and $\delta = \alpha^{-1}\beta^{-1}$. We also note that the infinite global attractor for $g(z) = z^2$ is $\{\odot, \alpha^{\beta^n}, \beta, \gamma^{\beta^n}, \delta^{\beta^n}, \beta^{\alpha\beta^n} \mid n \in \mathbb{Z}\}$

function μ_f can thus be phrased entirely in terms of group theory where the theory of self-similar groups comes into play.

We extend the virtual endomorphism ϕ to a map (not a homomorphism) $\hat{\phi} : G \rightarrow G$ by

$$\hat{\phi}(w) = \begin{cases} \phi(w) & \text{if } w \in \text{Dom}(\phi) \\ \phi(\alpha w) & \text{if } w \in \alpha^{-1}\text{Dom}(\phi). \end{cases}$$

This extension is motivated by the following lemma, which shows how ϕ transforms parabolic elements in terms of $\hat{\phi}$.

Lemma 6.1. *Let $x \in \{\alpha, \beta, \gamma\}$ and $w \in G$. For n minimal so that $\phi((x^n)^w)$ is defined, set $h = x^n$. Then*

$$\phi(h^w) = \begin{cases} h_1^{\hat{\phi}(w)} & \text{if } w \in \text{Dom}(\phi) \\ h_2^{\hat{\phi}(w)} & \text{if } w \in \alpha^{-1}\text{Dom}(\phi) \end{cases}$$

where h_1 and h_2 are the restrictions of h .

Proof. Let $h = \langle h_1, h_2 \rangle$ and suppose $w \in \text{Dom}(\phi)$. Then

$$\phi(h^w) = h_1^{\phi(w)}$$

If instead $w \notin \text{Dom}(\phi)$, then $w \in \alpha^{-1}\text{Dom}(\phi)$, so there exists $w' \in \text{Dom}(\phi)$ such that $w = \alpha^{-1}w'$. Then

$$\phi(h^w) = \phi(w'^{-1}\alpha h \alpha^{-1}w') = (\phi(\alpha h \alpha^{-1}))^{\phi(w')}$$

Recall that we chose to make the wreath recursion of α of the form $\alpha = \langle 1, \alpha|_2 \rangle \sigma$. Thus,

$$(\phi(\alpha h \alpha^{-1}))^{\phi(w')} = (\phi(\langle 1, \alpha|_2 \rangle \sigma \langle h_1, h_2 \rangle \langle \alpha|_2^{-1}, 1 \rangle \sigma))^{\phi(w')} = h_2^{\phi(w')}$$

□

The next lemma describes the global dynamics of $\hat{\phi}$. The simplicity of this statement depends heavily on our earlier choice that $\alpha|_1 = 1$ for every single map.

Lemma 6.2. *Let $w \in G$. There exists N such that for $n > N$, we have that*

$$\hat{\phi}^{\circ n}(w) \in \mathcal{N}$$

where \mathcal{N} is the (possibly infinite) nucleus of the self-similar action associated with ϕ .

Proof. If $w = \langle w_1, w_2 \rangle \in \text{Dom}(\phi)$, then $\hat{\phi}(w) = w_1$. If $w = \langle w_1, w_2 \rangle \sigma \in \alpha^{-1}\text{Dom}(\phi)$, then

$$\hat{\phi}(w) = \phi(\alpha w) = \phi(\langle 1, \alpha|_2 \rangle \sigma \langle w_1, w_2 \rangle \sigma) = w_2$$

Thus, $\hat{\phi}(w)$ is always equal to a restriction of w . □

Therefore, to determine the global dynamics of the pullback on curves, we need only examine the dynamics of ϕ on the set $\{(x^n)^w | x \in \{\alpha, \beta, \gamma\}, n \in \mathbb{Z}, w \in \mathcal{N}^{per}\}$ where \mathcal{N}^{per} is the set of periodic elements of $\hat{\phi}$ in \mathcal{N} . The results of these computations are given in Table 6.

For example, take the virtual endomorphism ϕ arising from the wreath recursion in the last row of Table 6:

$$\phi(\alpha^2) = \delta \quad \phi(\beta) = \alpha \quad \phi(\beta^\alpha) = 1$$

Checking the dynamics of $\hat{\phi}$ on \mathcal{N} , we see that all elements eventually map to the identity. So, we need only consider the powers of α , β , and δ (we use δ here in place of γ since it arises naturally in the computations with ϕ). Further, since the iteration under $\hat{\phi}$ always lands in the domain of ϕ , we need only consider the restriction under the first coordinate (i.e. the image under ϕ). Even powers of α are taken to powers of δ , powers of β are taken to powers of α , and even powers of δ are taken to powers of β . Thus, we have the finite global attractor as in Table 6.

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