COMBINATORIAL PROPERTIES OF NEWTON MAPS

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ABSTRACT. This paper presents a combinatorial model for postcritically finite Newton maps (rational maps that arise from applying Newton's method to complex polynomials). This model is a first step towards a combinatorial classification of postcritically finite Newton maps.

CONTENTS

1. INTRODUCTION

The dynamical properties of rational functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ have been intensely scrutinized over the last few decades, though in some ways the remarkable theory which has emerged is only in its early stages. Natural motivation for the topic comes from the study of Newton's method applied to a complex polynomial. For instance, it has long been observed that in some cases Newton's method does not converge to a root for open sets of initial values in $\mathbb{C}$; Smale posed the problem of "systematically finding" those polynomials whose Newton's method have such open sets. [Sma85, Problem 6]. In a different vein, a number of studies have been carried out on Newton's method as an algorithm [McM87, Sch].

Finite combinatorial models have been successfully created to encode the dynamics of postcritically finite complex polynomials [BFH92, Poi93], but similar attempts for rational maps have met with formidable difficulties (postcritically finite maps are chosen for study because they are structurally

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significant in parameter space, and because Thurston's characterization and rigidity theorem is available). This paper will produce a combinatorial invariant that will yield a classification of *all* postcritically finite Newton maps [LMS]. No other combinatorial classification of this scope exists for non-polynomial rational maps, as explicit classifications have only been made in the past for one-dimensional families.

Definition 1.1 (Newton map)**.** *A rational function* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *of degree* $d \geq 3$ *is called a* Newton map *if there is some complex polynomial* $p(z)$ *so that* $f(z) = z - \frac{p(z)}{p'(z)}$ *for all* $z \in \mathbb{C}$*.*

The *Newton map of* p is given by $N_p(z) = z - \frac{p(z)}{p'(z)}$, and it should be observed that N_p arises naturally when Newton's method is applied to find the roots of p. The cases $d < 3$ are excluded because they are trivial. Each root of p is an attracting fixed point of N_p, and the point at infinity is a repelling fixed point of N_p. The degree d coincides with the number of distinct roots of p. If N_p is postcritically finite, the finite fixed points of N_p must be superattracting, which implies that all roots of p are simple.

In this paper, we construct a finite forward invariant graph for N_p called an *extended Newton graph*. We then give an axiomatic definition of the class of graphs called "abstract extended Newton graphs" (see Definition 7.3) and show that our graphs satisfy these axioms. In [LMS], we show the converse: every abstract extended Newton graph comes from a postcritically finite Newton map. This leads to a combinatorial classification of postcritically finite Newton maps up to affine conjugacy in terms of abstract extended Newton graphs with an appropriate equivalence relation. Foundational to both articles will be the ideas in [MRS] which gives a classification of all postcritically *fixed* Newton maps, namely those Newton maps whose critical points are all mapped onto fixed points after finitely many iterations.

We give a brief overview of the graph invariant that is used to classify postcritically *finite* Newton maps, referring the reader to Figure 1 for motivation. If N_p is a postcritically finite Newton map, then as in [MRS], we define the channel diagram Δ of N_p to be the union of the accesses from finite fixed points of N_p to ∞ (see Section 3). Next, the Newton graph of level n is constructed to be the connected component of $N_p^{-n}(\Delta)$ containing ∞ and is denoted by Δ_n. For a sufficiently high level n, the Newton graph captures the behavior of critical points mapping onto fixed points.

We call a critical point *free* if it is not contained in the Newton graph Δ_n for any level n; put differently, a critical point is free if its forward orbit does not contain a fixed point. We now describe the combinatorial objects that capture the behavior of free critical points.

For each periodic postcritical point of N_p having period greater than one (i.e. a periodic postcritical point that isn't a root of p), we construct extended Hubbard trees (possibly degenerate, i.e. consisting of a single point) which contain them and describe the combinatorics of the corresponding polynomial-like maps (see Section 4). To capture the behavior of critical points that map into a Hubbard tree after some number of iterates, appropriate preimages are taken of these Hubbard trees.

Thus far, all postcritical points are contained in either the Newton graph or one of the Hubbard tree (preimages), but the Hubbard trees are disjoint from the Newton graph. To remedy this, "Newton rays" are used to connect the extended Hubbard trees to the Newton graph (see Section 5). The Newton rays are comprised of preimages of edges of the Newton graph, and the rays land at repelling periodic points on the Hubbard trees.

Now the *extended Newton graph*, denoted Δ_N^*, can be defined for N_p. It is a finite graph composed of:

- the Newton graph
- the Hubbard tree pieces for each free critical point of N_p
- Newton rays connecting each Hubbard tree piece to the Newton graph.

Restriction of N_p to Δ_N^* yields a self map, and the graph together with this self map is denoted (Δ_N^*, N_p).

The axioms for a *abstract extended Newton graph* are given in Definition 7.3, and the following theorem is proved.

Theorem 1.2 (Newton maps generate extended Newton graphs)**.** *For any extended Newton graph $\Delta_N^* \subset \widehat{\mathbb{C}}$ associated to a postcritically finite Newton map N_p, the pair (Δ_N^*, N_p) satisfies the axioms of an abstract extended Newton graph.*

It will be shown in [LMS] that every abstract extended Newton graph is realized by a unique postcritically finite Newton map up to affine conjugacy. This result will be used to establish a bijection between the set of postcritically finite Newton maps up to affine conjugacy and the set of abstract extended Newton graphs up to some explicit equivalence.

Structure of this paper: Section 2 introduces basic properties of Newton maps for later use, as well as a brief history of existing combinatorial models for Newton maps.

Section 3 constructs the Newton graph edges of the extended Newton graph. In so doing, the notions of a channel diagram, Newton graph and their abstract counterparts are defined. Extensions of certain graph maps to a branched cover of the two sphere is also discussed.

Section 4 initiates the construction of the Hubbard tree edges of the extended Newton graph. Preliminaries on extended and abstract extended Hubbard trees are covered in 4.1 and 4.2. The domains of renormalization are constructed in 4.3.

Section 5 initiates the construction of Newton ray edges, which will connect the Newton graph with fixed points of the polynomial-like mappings arising from renormalization. An ordering is placed on the rays to enable canonical choices among the rays landing at a single fixed point.

Section 6.1 combines the three types of edges to produce the extended Newton graph. An example of such a graph is given in 6.2.

Section 7 defines the abstract analog of Newton rays and extended Newton graphs, and shows that an extended Newton graph constructed for a postcritically finite Newton maps satisfies the abstract definition. The main

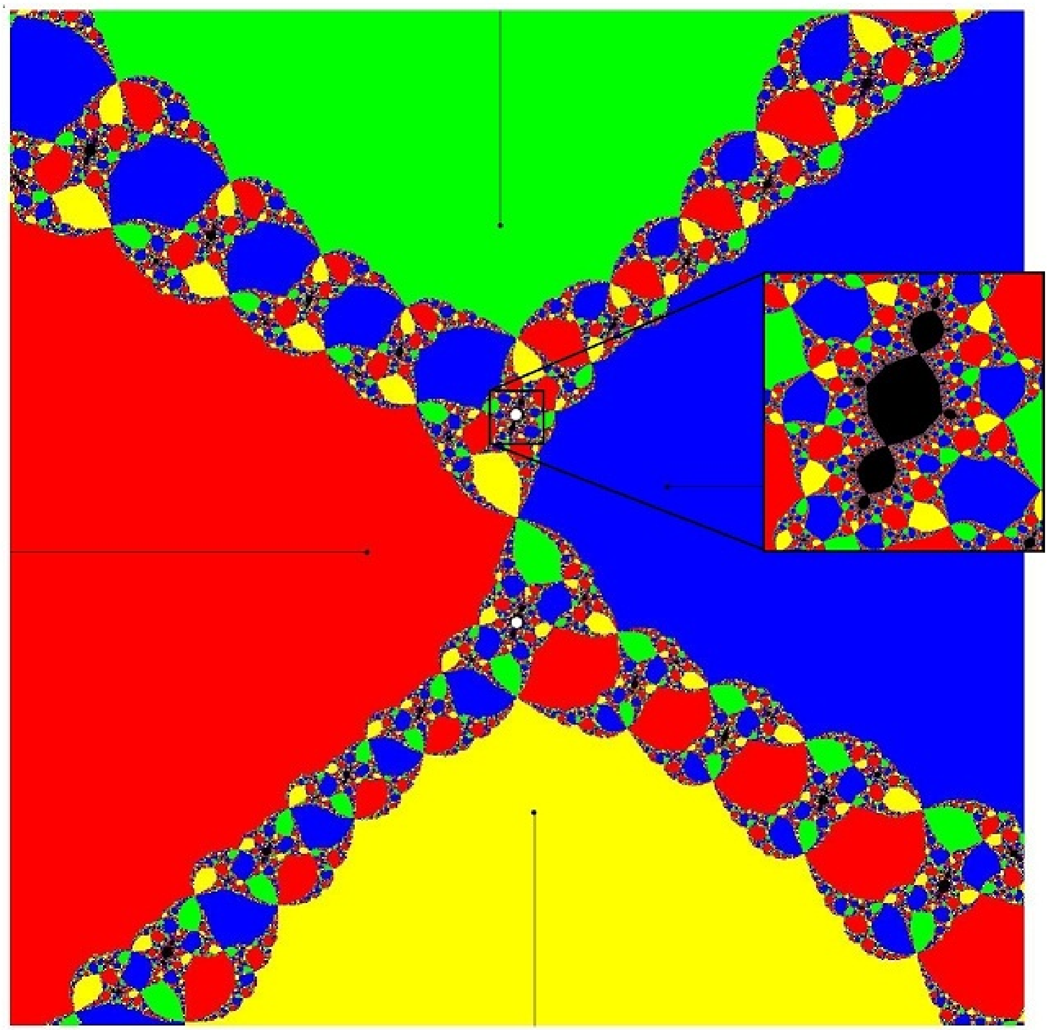

FIGURE 1. Part of the dynamical plane of the Newton map of degree 4 (with an inset zoom) for the monic polynomial with roots given approximately by $\pm(.593 + .130i)$ and $\pm(-.0665 + 1.157i)$. Four black dots represent these roots which are fixed critical points of the Newton map, and their basins are indicated by different colors. The two white dots represent the free critical points at $\pm.408i$ (both having disjoint periodic forward orbits of length four, though the two orbits lie in the same four "little basilicas"). The basins of the free critical points are black, and correspond to open sets of starting points of Newton's method that do not converge to roots. The channel diagram has four edges (drawn up to homotopy by thin black lines), each corresponding to rays connecting the roots to infinity through their respective immediate basins.

result of the paper (Theorem 1.2) is proven.

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2. Known results about Newton maps

This section will catalog some well-known properties of Newton maps for later use. A brief history of the various combinatorial models for Newton maps will be given as well.

The following result is crucial to this study, and is a special case of Proposition 2.8 in [RS07].

Proposition 2.1 (Head's theorem)**.** [Hea87, Proposition 2.1.2] *A rational map f of degree $d \geq 3$ is a Newton map if and only if ∞ is a repelling fixed point of f and for each of the other fixed points $\xi \in \widehat{\mathbb{C}}$, there is an integer $m \geq 1$ so that $f'(\xi) = (m-1)/m$.*

Let $p(z)$ be a monic polynomial of degree d with complex coefficients and simple roots $a_1, a_2, ..., a_d$. Define the Newton map corresponding to p by

$$N_p(z) = z - \frac{p(z)}{p'(z)}. \tag{1}$$

One can see from the equation

$$N_p'(z) = \frac{p(z)p''(z)}{(p'(z))^2}.$$

that the roots of p are superattracting fixed points of $N_p(z)$. The point at infinity is a repelling fixed point of N_p with multiplier $d/(d-1)$.

Note that the roots of p must be simple for the purposes of this study because otherwise the corresponding Newton map would have an attracting fixed point that is not superattracting, and would thus not be postcritically finite. The map N_p has degree d, and its $d+1$ fixed points are given by the $a_1, a_2, ..., a_d, \infty$; thus all finite fixed points of the Newton map are critical.

Shishikura [Shi09] proved that the Julia set of a rational map is connected if there is only one repelling fixed point. Combining this with the facts just mentioned, he obtains the following.

Proposition 2.2. *The Julia set of a Newton map N_p is connected.*

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation-preserving branched cover of degree greater than one. Denote the local degree of f at a point z by $\deg_z f$.

Definition 2.3. *Set $C_f = \{\text{critical points of } f\} = \{x | \deg_x f > 1\}$ and*

$$P_f = \bigcup_{n \geq 1} f^n(C_f).$$

The map f is called a postcritically finite branched cover *if P_f is finite. We say that f is* postcritically fixed *if for each $x \in C_f$, there exists $N \in \mathbb{N}$ such that $f^{\circ N}(x)$ is a fixed point of f.*

Definition 2.4 (Immediate basin)**.** *Let N_p be a Newton map and $\xi \in \mathbb{C}$ a finite fixed point of N_p. Let $B_\xi = \{z \in \mathbb{C} : \lim_{n\to\infty} N_p^n(z) = \xi\}$ be the basin (of attraction) of ξ. The connected component of B_ξ containing ξ is called the* immediate basin *of ξ and denoted U_ξ.*

It was shown by Przytycki that U_ξ is simply connected and unbounded [Prz89]. This result was strengthened by Shishikura who showed that every component of the Fatou set is simply connected, not just immediate basins [Shi09].

Definition 2.5 (Invariant access to ∞)**.** *Let ξ be an attracting fixed point of N_p and U_ξ its immediate basin. An* access *of ξ to ∞ is a homotopy class of curves in U_ξ that begin at ξ, land at ∞ and are homotopic in U_ξ with fixed endpoints.*

The following proposition is used to produce the first-level combinatorial data for Newton maps.

Let m_ξ be the number of critical points of a Newton map N_p in the immediate basin U_ξ, counted with multiplicity. Then $N_p|_{U_\xi}$ is a branched cover of degree $m_\xi + 1$.

Proposition 2.6 (Accesses to infinity in immediate basins)**.** [HSS01] *The immediate basin U_ξ has exactly m_ξ accesses to ∞.*

Combinatorial models for various types of postcritically finite Newton maps exist. Janet Head introduced the "Newton tree" to characterize postcritically finite cubic Newton maps [Hea87]. Tan Lei built upon these ideas to give a classification of postcritically finite cubic Newton maps in terms of matings and captures [Tan97]. Tan Lei also gave another combinatorial classification of the Newton cubic family using abstract graphs. More precisely, every postcritically finite cubic Newton map gives rise to a forward invariant finite connected graph that satisfies certain axioms. Conversely, every graph which satisfies these axioms is realized by a unique postcritically finite cubic Newton map using Thurston's theorem. Finally, the graph associated to a postcritically finite cubic Newton map is realized by the same cubic Newton map under Thurston's theorem (all graphs and rational maps are considered up to the natural equivalences).

Fewer results exist for higher degree. Jiaqi Luo studies Newton maps of arbitrary degree with exactly one non-fixed critical value, which we call "unicritical Newton maps". For such maps, Luo constructs a forward-invariant, finite topological graph analogous to the Newton graph of this paper. In the spirit of Tan Lei's work, he defines a "topological Newton map" to be a branched cover with the same critical orbit properties as a unicritical Newton map, and then shows that Thurston obstructions for topological Newton maps may only be Levy cycles of a special type [Luo94]. Assuming further that a topological Newton map satisfies certain explicit conditions on the attracting basins of the fixed critical points, Luo proves that no Thurston obstructions exist if the non-fixed critical value is either periodic or contains a fixed critical point in its orbit [Luo93]. In a different vein, and using different methods, [CGN+13] describes a process by which Newton maps whose critical points are all fixed may be produced by "blowing up" the edges of a multigraph.

In his thesis, Johannes Rückert [Rüc06] classified all *postcritically fixed* Newton maps for arbitrary degree (the results are also found in [MRS]). A Newton map is called *postcritically fixed* if all its critical points are mapped onto fixed points after finitely many iterations. For every postcritically fixed Newton map, a connected forward-invariant finite graph that contains the whole postcritical set is constructed. The notion of an "abstract Newton graph" is introduced, and it is seen that the forward-invariant graph just described is in fact an abstract Newton graph. It is shown that each abstract Newton graph is realized by a unique postcritically fixed Newton maps, and that the abstract graphs give the classification.

3. Newton graphs from Newton maps

Some preliminaries about graph maps are presented, following [BFH92, Chapter 6]. In particular, a condition under which a graph map may be uniquely extended to a branched cover of the whole sphere is presented which will be useful for the definition of the abstract extended Newton graph. The following is the so-called "Alexander trick" which is fundamental to such extension results.

Lemma 3.1. *Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism. Then there exists an orientation preserving homeomorphism $\overline{h} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $\overline{h}|_{\mathbb{S}} = h$. The map $\overline{h}$ is unique up to isotopy relative $\mathbb{S}^1$.*

Definition 3.2 (Finite graph)**.** *A* finite graph Γ *is the quotient of a finite disjoint union of* edges *(abstract spaces homeomorphic to closed intervals) by an equivalence relation on the set of endpoints. Each equivalence class is called a* vertex *of the graph. A* finite embedded graph *is a homeomorphic image of a finite graph into $\mathbb{S}^2$.*

We assume in the following that all graphs are embedded in $\mathbb{S}^2$.

Definition 3.3 (Graph map)**.** *Let Γ_1, Γ_2 be connected finite graphs. A map $f : \Gamma_1 \to \Gamma_2$ is called a* graph map *if it is continuous and injective on each edge of the graph Γ_1, and the forward and backward images of vertices are vertices.*

Remark 3.4 (Notation and terminology). We will define the extended Newton graph Δ_N^* equipped with a "graph map" $\Delta_N^* \to \Delta_N^*$ given by the restriction of the Newton map N_p (Definition 6.3). Strictly speaking, this restriction is not a graph map since Newton ray edges contain preimages of vertices in the Newton graph that are not counted as vertices in Δ_N^* (these vertices were purposely ignored since we seek to produce a finite graph). However, after adding this finite collection of vertices to Δ_N^*, we obtain a new graph $(\Delta_N^*)^+$ where clearly $N_p : (\Delta_N^*)^+ \to \Delta_N^*$ is an honest graph map. This procedure can always be done, and so we prefer to abuse notation slightly by calling the restriction of N_p a graph map.

Definition 3.5 (Regular extension)**.** *Let $f : \Gamma_1 \to \Gamma_2$ be a graph map. An orientation-preserving branched cover $\overline{f} : \mathbb{S}^2 \to \mathbb{S}^2$ is called a* regular extension *of f if $\overline{f}|_{\Gamma_1} = f$ and $\overline{f}$ is injective on each component of $\mathbb{S}^2 \setminus \Gamma_1$.*

It follows that every regular extension $\overline{f}$ may have critical points only at the vertices of Γ_1.

Lemma 3.6 (Isotopic graph maps)**.** [BFH92, Corollary 6.3] *Let $f, g : \Gamma_1 \to \Gamma_2$ be two graph maps that coincide on the vertices of Γ_1 such that for each edge $e \subset \Gamma_1$ we have $f(e) = g(e)$ as a set. Suppose that f and g have regular extensions $\overline{f}, \overline{g} : \mathbb{S}^2 \to \mathbb{S}^2$. Then there exists a homeomorphism $\psi : \mathbb{S}^2 \to \mathbb{S}^2$, isotopic to the identity relative the vertices of Γ_1, such that $\overline{f} = \overline{g} \circ \psi$.*

We must establish some notation for the following proposition from [BFH92]. Let $f : \Gamma_1 \to \Gamma_2$ be a graph map. For each vertex v of Γ_1, choose a neighborhood $U_v \subset \mathbb{S}^2$ such that all edges of Γ_1 that enter U_v terminate at v, the vertex v is the only vertex in U_v, and the neighborhoods U_v and U_w are disjoint for $v \neq w$; we may assume without loss of generality that in local cordinates, U_v is a round disk of radius 1 that is centered at v, that the intersection of any edge with U_v is either empty or a radial line segment, and that $f|_{U_v}$ is length-preserving. Make analogous assumptions for Γ_2.

We describe how to explicitly extend f to each U_v. For a vertex $v \in \Gamma_1$, let γ_1 and γ_2 be two adjacent edges ending there. In local coordinates, these are radial lines at angles Θ_1, Θ_2 where $0 < \Theta_2 - \Theta_1 \leq 2\pi$ (if v is an endpoint of Γ_1, then set $\Theta_1 = 0$, $\Theta_2 = 2\pi$). In the same way, choose arguments Θ_1', Θ_2' for the image edges in $U_{f(v)}$ and extend f to a map $\tilde{f}$ on $\Gamma_1 \cup \bigcup_v U_v$ by setting

$$(\rho, \Theta) \mapsto \left(\rho, \frac{\Theta_2' - \Theta_1'}{\Theta_2 - \Theta_1} \cdot \Theta\right),$$

where (ρ, Θ) are polar coordinates in the sector bounded by the rays at angles Θ_1 and Θ_2. In particular, sectors are mapped onto sectors in an orientation-preserving way.

Proposition 3.7. [BFH92, Proposition 5.4] *A graph map $f : \Gamma_1 \to \Gamma_2$ has a regular extension if and only if for every vertex $y \in \Gamma_2$ and every component U of $\mathbb{S}^2 \setminus \Gamma_1$, the extension $\tilde{f}$ is injective on*

$$\bigcup_{v \in f^{-1}(y)} U_v \cap U .$$

The combinatorial classification of postcritically *fixed* Newton maps (all critical points mapping onto fixed points after finitely many iterations) was given in [MRS] using a combinatorial object called the "Newton graph". We give the analogous construction for a postcritically *finite* Newton map, noting that the results mentioned below from [MRS] hold in this more general context. The graph constructed below will also be called the Newton graph.

The extended Newton graph that we associate to a Newton map is a finite graph Δ_N^* equipped with a self-map coming from the restriction of N_p (Definition 6.3). This restriction is not a graph map in general since Newton ray edges contain finitely many preimages of vertices in the Newton graph that are not vertices in Δ_N^*. This motivates the following weaker definition where the condition on preimages of vertices has been dropped.

Definition 3.8 (Weak graph map)**.** *A continuous map $f : \Gamma_1 \to \Gamma_2$ is called a* weak graph map *if it is injective on each edge of the graph Γ_1 and the forward image of vertices are vertices.*

Remark 3.9. Given a weak graph map $f : \Gamma_1 \to \Gamma_2$, the combinatorics of the domain can be slightly altered to produce a graph map $\hat{f} : \hat{\Gamma}_1 \to \Gamma_2$ in the following way. We take the graph $\hat{\Gamma}_1$ to have vertices given by $f^{-1}(\Gamma_2')$, and edges given by the closures of complementary componenents of $\Gamma_1 \setminus f^{-1}(\Gamma_2')$. We simply take $\hat{f} = f$.

Let the superattracting fixed points of a postcritically finite Newton map N_p be denoted by $a_1, a_2, \ldots, a_d$. Let U_i denote the immediate basin of a_i. Then U_i has a global Böttcher coordinate $\phi_i : (\mathbb{D}, 0) \to (U_i, a_i)$ with the property that $N_p(\phi_i(z)) = \phi_i(z^{k_i})$ for each $z \in \mathbb{D}$ (the complex unit disk), where $k_i - 1 \geq 1$ is the multiplicity of a_i as a critical point of N_p. The map $z \to z^{k_i}$ fixes $k_i - 1$ internal rays in $\mathbb{D}$. Under ϕ_i, these map to $k_i - 1$ pairwise disjoint (except for endpoints) simple curves $\Gamma_i^1, \Gamma_i^2, \ldots, \Gamma_i^{k_i-1} \subset U_i$ that connect a_i to ∞, are pairwise non-homotopic in U_i (with homotopies fixing the endpoints) and are invariant under N_p. They represent all accesses to ∞ of U_i (see Proposition 2.6). The union

$$\Delta = \bigcup_i \bigcup_{j=1}^{k_i-1} \overline{\Gamma_i^j}$$

forms a connected graph in $\widehat{\mathbb{C}}$ that is called the *channel diagram*. It follows from the definition that $N_p(\Delta) = \Delta$. The channel diagram records the mutual locations of the immediate basins of N_p and provides a first-level combinatorial information about the dynamics of the Newton map. For any $n \geq 0$, denote by Δ_n the connected component of $N_p^{-n}(\Delta)$ that contains Δ. The pair $(\Delta_n, N_p|_{\Delta_n})$ is called the *Newton graph* of N_p at level n. The following result is proven in [MRS].

Theorem 3.10. [MRS, Theorem 3.4] *There exists a positive integer N so that Δ_N contains all poles of N_p.*

An obvious corollary is that for any prepole, there exists sufficiently large m such that this prepole is in Δ_m. Another corollary that will be used later is the following.

Corollary 3.11. *Let k and m be integers so that $k - m \geq N$ where the integer N is the minimal level of the Newton graph so that Δ_N contains all poles of N_p. Then $N_p^{-m}(\Delta_k) = \Delta_{m+k}$.*

The following theorem gives structure to the basins of attraction of finite fixed points of N_p.

Theorem 3.12. [MRS, Theorem 1.4] *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Newton map with attracting fixed points $a_1, \ldots, a_d \in \mathbb{C}$, and let U_0' be a component of some B_i, the basin of attraction of a_i. Then, U_0' can be connected to ∞ by the closures of finitely many components $U_1', \ldots, U_k'$ of $\bigcup_{i=1}^d B_i$. More precisely, there exists a curve $\gamma : [0,1] \to \widehat{\mathbb{C}}$ such that $\gamma(0) = \infty$, $\gamma(1) \in U_0'$ and for every $t \in [0,1]$, there exists $m \in \{0, 1, \ldots, k\}$ such that $\gamma(t) \in \overline{U}_m'$.*

The immediate goal of [MRS] was to give a classification of postcritically fixed Newton maps in terms of abstract Newton graphs. However, along the way it was shown that the Newton graph of postcritically *finite* Newton maps also satisfy the axioms. The pertinent definitions and theorem are presented here.

Definition 3.13 (Abstract channel diagram)**.** *An* abstract channel diagram *of degree $d \geq 3$ is a graph $\Delta \subset \mathbb{S}^2$ with vertices $v_\infty, v_1, \ldots, v_d$ and edges $e_1, \ldots, e_l$ that satisfies the following properties:*

(1) $l \leq 2d-2$;
(2) each edge joins v_∞ to some v_i for $i \in \{1, ..., d\}$;
(3) each v_i is connected to v_∞ by at least one edge;
(4) if e_i and e_j both join v_∞ to v_k, then each connected component of $\mathbb{S}^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex of Δ.

It is not difficult to check that the channel diagram Δ constructed for a Newton map N_p above satisfies conditions of Definition 3.13. Indeed by construction, Δ has at most $2d-2$ edges and it satisfies (2) and (3). Finally, Δ satisfies (4), because for any immediate basin U_ξ of N_p, every component of $\mathbb{C} \setminus U_\xi$ contains at least one fixed point of N_p [RS07, Corollary 5.2].

Definition 3.14 (Abstract Newton graph)**.** *Let $\Gamma \subset \mathbb{S}^2$ be a connected finite graph, Γ' the set of its vertices and $f : \Gamma \to \Gamma$ a graph map. The pair (Γ, f) is called an* abstract Newton graph of level N_Γ *if it satisfies the following conditions:*

(1) There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that f fixes each vertex and each edge (pointwise) of Δ.
(2) There is an extension of the graph map f to a branched cover $\tilde{f} : \mathbb{S}^2 \to \mathbb{S}^2$ such that the following conditions (3) – (6) are satisfied.
(3) Γ equals the component of $\tilde{f}^{-N_\Gamma}(\Delta)$ that contains Δ.
(4) If $v_\infty, v_1, \ldots, v_{d_\Gamma}$ are the vertices of Δ, then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq \infty$. Moreover, there are exactly $\deg_{v_i}(\tilde{f}) - 1 \geq 1$ edges in Δ that connect v_i to v_∞ for $i \neq \infty$, where $\deg_x(\tilde{f})$ denotes the local degree of $\tilde{f}$ at $x \in \Gamma'$.
(5) $\sum_{x \in \Gamma'} \Big(\deg_x(\tilde{f}) - 1\Big) \leq 2d_\Gamma - 2$.
(6) The graph $\overline{\Gamma \setminus \Delta}$ is connected.

Note that the extension in Condition (2) is not assumed to be regular, as there may be critical points of $\tilde{f}$ that are not in Γ (in contrast to the definition of abstract Newton graph given in [MRS]). This difference corresponds to the fact that in our setting it is possible that the forward orbit of a critical point does not intersect the channel diagram.

It follows from [MRS] that if N_p is a postcritically finite Newton map, then the pair (Δ_N, N_p) satisfies all conditions of an abstract Newton graph (Definition 3.14), where N is chosen to be the minimal positive integer such that any non-free critical point is mapped by $N_p^{\circ(N-1)}$ into the channel diagram Δ. The following is a restatement of Theorem 1.5 from [MRS] adapted to the more general setting of postcritically *finite* maps.

Theorem 3.15. *For every postcritically finite Newton map N_p, there exists some minimal level N so that (Δ_k, N_p) is an abstract Newton graph of level k for all $k \geq N$.*

Note that the level N in this theorem is not necessary the level of the Newton graph chosen in the construction of the extended Newton graph (see Definition 6.1) though it does give a lower bound.

4. Hubbard trees from Newton maps

In this section we describe well-known results about Hubbard trees and polynomial-like maps, and then apply these results to model the dynamics about non-fixed postcritical points of Newton maps.

4.1. **Extended Hubbard trees.** Douady and Hubbard [DH85a] showed how to extract from any postcritically finite polynomial a combinatorial invariant called the Hubbard tree, and it was shown that such trees distinguish inequivalent polynomials. The complete classification of postcritically finite polynomials in terms of Hubbard trees is given in [Poi93].

A *tree* is a topological space which is uniquely arcwise connected and homeomorphic to a union of finitely many copies of the closed unit interval. All trees are assumed to be embedded in $\mathbb{S}^2$.

Let f be a complex polynomial. Define the *filled Julia set* $K(f)$ to be the set of $z \in \mathbb{C}$ whose forward orbit under f is bounded. The *Julia set* $J(f)$ is the boundary of $K(f)$.

We recall some facts about the dynamics of postcritically finite polynomials [Mil06]. For each Fatou component U_i, there is exactly one point $x \in U_i$ such that $f^n(x) \in P_f$ for some non-negative integer n. We call x the *center* of U_i. Denote by U_{i+1} the Fatou component containing $f(x)$. A classical theorem of Böttcher implies that there are holomorphic isomorphisms $\phi_i : (\mathbb{D}, 0) \to (U_i, x)$ and $\phi_{i+1} : (\mathbb{D}, 0) \to (U_{i+1}, f(x))$ such that for all $z \in \mathbb{D}$:

$$\phi_{i+1}(z^{k_i}) = f(\phi_i(z)),$$

where k_i is the local degree degree of f near x. If f is a postcritically finite polynomial, then the Julia set $J(f)$ is a connected and locally connected compact set [DH85a]. Since each Fatou component has locally connected boundary, Caratheodory's theorem implies that the map ϕ_i extends continuously to the unit circle. Let $R(t) = \{r \exp(2\pi i t) | 0 \leq r \leq 1\}$. The image $R_i(t) = \phi_i(R(t))$ is called the *ray of angle t in U_i*. If $x = \infty$, the ray $R_i(t)$ is called an *external ray*, otherwise it is called *internal ray*.

We now describe the construction of Hubbard trees following the second chapter of [DH85a]. A Jordan arc $\gamma \subset K(f)$ is called *allowable* if for every Fatou component U_i, the set $\phi_i^{-1}(\gamma \cap \overline{U_i})$ is contained in the union of two rays of $\overline{\mathbb{D}}$. For every z, z' in $K(f)$ there is a unique allowable arc joining them [DH85a, Proposition 2.6]. We denote this arc by $[z, z']_{K(f)}$. We say that a subset $X \subset K(f)$ is *allowably connected* if for every $z_1, z_2 \in X$ we have $[z_1, z_2]_{K(f)} \subset X$. The intersection of a family of allowably connected subsets is allowably connected. The *allowable hull* $[X]_K$ of $X \subset K(f)$ is defined to be the intersection of all the allowably connected subsets of $K(f)$ containing X. If X is a finite set, then the allowable hull $[X]_K$ is a topological finite tree [DH85a, Proposition 2.7].

In the following definition ([Poi93, Definition I.1.9]), C_f denotes the set of critical points.

Definition 4.1 (Hubbard tree)**.** *Let M be a finite forward invariant set with $C_f \subset M$. The Hubbard tree $H(M)$ is the tree generated by M, i.e. the allowable hull $[M]_K$.*

Typically $M = P_f$ in the literature. We will always wish to include other points as discussed below.

These Hubbard trees (including those with additional marked points) are axiomatized as *abstract Hubbard trees* in Section II.4 of [Poi93] (see also [Poi10]). Poirier assigns a degree to each Hubbard tree in terms of local degree of the tree dynamics (he always assumes that the degree is greater than one). Under a natural partial ordering on abstract Hubbard trees, Poirier shows that there is a unique minimal abstract Hubbard tree that is in fact the tree generated by the orbit of C_f [Poi93, Proposition II.4.5]. An equivalence relation on abstract Hubbard trees is given in [Poi93, Definition II.4.3], where two trees are equivalent if they are homeomorphic and the dynamics are respected (among other things, this means the dynamics on vertices are conjugate).

We now state two essential theorems, begining with the crucial realization theorem for Hubbard trees [Poi93, Theorem II.4.7].

Theorem 4.2 (Realization of abstract Hubbard trees)**.** *For any abstract Hubbard tree H, there exists a unique (up to affine conjugacy) postcritically finite polynomial f such that $H(M)$ is equivalent to H where $M \supset C_f$ is a finite forward-invariant set.*

Next we give the classification theorem which is essentially a consequence of the construction of Hubbard trees, the realization theorem, and minimality [Poi93, Theorem II.4.8].

Theorem 4.3 (Classification of postcritically finite polynomials)**.** *The set of affine conjugacy classes of postcritically finite polynomials of degree at least two is in bijective correspondence with the set of equivalence classes of minimal abstract Hubbard trees.*

We must now give an analogous exposition for polynomials where all cycles up to a certain length are marked. The fixed points are marked because we will use Newton rays to connect repelling fixed points in Hubbard trees to the Newton graph. It is also necessary at times to mark cycles of longer length so that we may maintain combinatorial control of the free critical points of the Newton map that map into repelling cycles of filled Julia sets.

The set of marked points for the polynomial f including cycles of length n or less is denoted

$$M_n = P_f \cup \bigcup_{i=0}^{n} \{z | f^{\circ i}(z) = z\}.$$

Definition 4.4 (Extended Hubbard tree)**.** *An* extended Hubbard tree *is a Hubbard tree of the form $H(M_n)$ where $n \geq 1$. We say that $H(M_n)$ has* cycle type *n.*

Remark 4.5. If f is a degree one map with a unique finite repelling fixed point $z_0 \in \mathbb{C}$, the extended Hubbard tree $H(M_n)$ consists of the point z_0

equipped with the identity map for all n (see Definition 4.1). If an extended Hubbard tree consists of a single point, it is said to be *degenerate*. This occurs only when f has degree one.

As mentioned, the definition of abstract Hubbard tree (Section II.4 [Poi93]) allows for marked points beyond the postcritical set, and there is a well-defined notion of degree for abstract Hubbard trees.

Definition 4.6 (Abstract extended Hubbard tree)**.** *An* abstract extended Hubbard tree *(of degree greater than one) is an abstract Hubbard tree H whose vertex set includes $deg(H)^k$ cycles of length k for each $1 \leq k \leq n$. Such a tree is said to have cycle type n. An abstract extended Hubbard tree of degree one consists of a single point with self map given by the identity and is said to have* cycle type one*.*

The partial order on abstract Hubbard trees defined by [Poi93, Definition II.4.2] induces an order on abstract extended Hubbard trees of fixed degree and fixed cycle type n. In analogy to [Poi93, Proposition II.4.5], we conclude that there is a unique minimal abstract extended Hubbard tree under this partial order, namely the tree generated by the points in cycles of length n or less. By convention, the minimal degree one abstract extended Hubbard tree is the degenerate Hubbard tree.

Since each extended abstract Hubbard tree is in fact an abstract Hubbard tree (except in degree one where realization is evident anyway), we may apply our Theorem 4.2 (the realization theorem) to abstract extended Hubbard trees.

4.2. **Polynomial-like maps and renormalization.** Polynomial-like maps were introduced by Douady and Hubbard [DH85b] and have played an important role in complex dynamics ever since. They will be used in Section 4.3 to model the dynamics close to critical points whose orbit does not intersect the channel diagram.

Definition 4.7. *A* polynomial-like *map of degree $d \geq 1$ is a triple (f, U, V) where U, V are topological disks in $\mathbb{C}$, the set $\overline{U}$ is a compact subset of V, and $f : U \to V$ is a proper holomorphic map such that every point in V has d preimages in U when counted with multiplicities.*

Remark 4.8. The above definition differs slightly from the typical one found in the literature, as we allow that $d = 1$. Such a map is called a *degenerate polynomial-like map.* The following two theorems are stated in their original sources for $d \geq 2$, but we include the $d = 1$ case without justification, as the proof in this case is trivial.

Definition 4.9. *Let $f : U \to V$ be a polynomial-like map. The* filled Julia set *of f is the set of points in U that never leave V under iteration of f, i.e.*

$$K(f) = \bigcap_{n=1}^{\infty} f^{-n}(V).$$

As with polynomials, we define the Julia set *as $J(f) = \partial K(f)$.*

The simplest example of polynomial-like maps is a restriction of any polynomial: for a polynomial p of degree $d \geq 2$, let $V = \{z \in \mathbb{C} : |z| < R\}$ for sufficiently large R and $U = f^{-1}(V)$. Then $p : U \to V$ is a polynomial-like mapping of degree d.

Remark 4.10. In general, for a triple (f, U, V) with $U \subset V$ and $f : U \to V$ a proper holomorphic map we denote by

$$K(f, U, V) = \bigcap_{n=1}^{\infty} f^{-n}(V)$$

the set of points in U that never leave V under iteration by f.

Two polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy ψ between f and g that is defined on a neighborhood of their respective filled Julia sets so that $\bar{\partial}\psi = 0$ on $K(f)$.

The crucial relation between polynomial-like maps and polynomials is explained in the following theorem, due to Douady and Hubbard [DH85b].

Theorem 4.11 (The straightening theorem)**.** *Let $f : U' \to U$ be a polynomial-like map of degree d. Then f is hybrid equivalent to a polynomial P of degree d. Moreover, if $K(f)$ is connected and $d \geq 2$, then P is unique up to affine conjugation.*

Now we define the notion of renormalization of rational functions. Let R be a rational function of degree d and let z_0 be a critical or postcritical point of R.

Definition 4.12. *R^n is called* renormalizable about z_0 *if there exist open disks $U, V \subset \mathbb{C}$ satisfying the following conditions:*

(1) $z_0 \in U$.
(2) (R^n, U, V) is a polynomial-like map with connected filled Julia set.

A *renormalization* is a polynomial-like restriction $\rho = (R^n, U, V)$ as just described. We call n the *period* of the renormalization ρ. The filled Julia set of ρ is denoted by $K(\rho)$, the Julia set $J(\rho)$, and the critical and postcritical sets by $C(\rho)$ and $P(\rho)$ respectively. The *i'th small filled Julia set* is given by $K(\rho, i) = R^i(K(\rho))$ and the *i'th Julia set* by $J(\rho, i) = R^i(J(\rho))$. The *i'th small critical set* is $C(\rho, i) = K(\rho, i) \cap C_R$, and *i'th small postcritical set* is $P(\rho, i) = K(\rho, i) \cap P_R$.

The following uniqueness result is proven for degree greater than 1 in [McM94, Theorem 7.1].

Theorem 4.13 (Uniqueness of renormalization)**.** *Let $\rho = (R^n, U, V)$ and $\rho' = (R^n, U', V')$ be two renormalizations of the same period. If the degree of ρ is greater than 1, and $C(\rho, i) = C(\rho', i)$ for all $1 \leq i \leq n$, then the filled Julia sets are the same, i.e. $K(\rho) = K(\rho')$. If the degree of ρ equals 1 and $P(\rho, i) = P(\rho', i)$ for all $1 \leq i \leq n$, the filled Julia sets are the same.*

In Section 4.1 the notion of extended Hubbard trees for a given postcritically finite polynomials was introduced. Note that the same construction applies to polynomial-like maps $f : U' \to U$ with connected filled Julia set. We use this in the following.

4.3. **Renormalization of Newton maps.** The Newton graph Δ_N of N_p defined in Section 3 divides the complex plane into finitely many pieces, and each free critical point has an itinerary with respect to this partition. In this section we associate a polynomial-like map to each periodic free postcritical point (or equivalently, each periodic postcritical point of period two or more). Each polynomial-like map has an associated extended Hubbard tree, yielding a combinatorial model of the dynamics in some neighborhood of these postcritical points. These Hubbard trees comprise one piece of the combinatorial model for postcritically finite Newton maps.

First, Lemma 4.14 associates to each periodic postcritical point z_k a triple $(N_p^{m(k)}, U_k, V_k)$ which very much resembles a polynomial-like map. One difference is that the containment $U_k \subset V_k$ is not necessarily compact (a matter resolved by slight perturbations in Lemma 4.19). Still, in this context it is possible to define an analogue of the filled Julia set using Remark 4.10.

Lemma 4.14 (Construction of unperturbed maps)**.** *For a postcritically finite Newton map N_p there exist positive integers M, N' and domains (U_k, V_k), $1 \le k \le M$ with $U_k \subset V_k$ satisfying the following properties:*

1. *Any two domains U_k, U_l, $k \neq l$ are pairwise disjoint, and every non-fixed periodic postcritical point of N_p is in some U_k.*
2. *For every (U_k, V_k) there exists $m(k) \in \mathbb{Z}^+$ such that $N_p^{m(k)} : U_k \to V_k$ is a proper map (whose degree is denoted $d_k \ge 1$).*
3. *Each of U_k and V_k are complementary components of the Newton graph at different levels, and $\partial V_k \subset \Delta_{N'}$.*
4. *The postcritical points of N_p that are in V_k are contained in the set $K(N_p^{m(k)}, U_k, V_k)$.*

Proof. Let N be the level of the Newton graph at which Δ_N contains all poles (guaranteed by Theorem 3.10). Let z_1 be a postcritical point of period at least two. Since z_1 is not contained in Δ_N, it must lie in some complementary component of Δ_N which we denote V_1'.

Let $m(1)$ be the smallest integer multiple of the period of z_1 greater than or equal to N. For convenience, let $m := m(1)$. Let V_1'' be the unique preimage of V_1' under N_p^m such that $z_1 \in V_1''$. Observe that $V_1'' \subset V_1'$ since Δ_N is forward invariant. Furthermore, V_1'' must be a complementary component of Δ_{N+m} by Corollary 3.11. Denote by $F : V_1'' \to V_1'$ the restriction of N_p^m to V_1''.

We now construct V_1. Since N_p is postcritically finite, there exists a positive integer n such that $P_F \cap F^{-n}(V_1') \subset K(F, V_1'', V_1')$. Among such integers n choose the minimal one and denote it by $n(1)$. Let V_1 be the component of $F^{-n(1)}(V_1')$ that contains z_1. By construction of V_1, every point in $P_{N_p} \cap V_1$ has a finite F-orbit lying in V_1 (this is needed to prove part (4) of the lemma). Also, arguing as before, V_1 is a subset of V_1'' and a complementary component of Δ_{N_1} with $N_1 := N + n(1)m(1)$.

We now construct U_1. Arguing as before, the component of $F^{-1}(V_1)$ that contains z_1 is a subset of V_1 and a complementary component of the Newton graph of level $N_1 + m(1)$. Let U_1 be this component.

Now consider the graph Δ_{N_1} instead of Δ_N and some other periodic postcritical point z_2 of period at least two which doesn't lie in V_1. Carry out a similar procedure to the previous three paragraphs to construct the domains (U_2, V_2). Again, denote by N_2 the minimal level of the Newton graph that contains the boundary of V_2. We analogously construct the required set of domains (U_k, V_k), $1 \leq k \leq M$, (where M is some integer less than or equal to the number of periodic points in P_{N_p}) and a level of the Newton graph that contains the boundaries of all components V_k for $1 \leq k \leq M$ is given by

$$N' = N + \sum_{k=1}^{M} n(k)m(k).$$

□

Define the restriction $F_k := N_p^{m(k)}|_{U_k}$ and observe that $F_k : U_k \to V_k$. The rest of this section is devoted to showing that the sets U_k and V_k can be modified so that the restriction of $N_p^{m(k)}$ to this new domain is a polynomial-like map, and showing that the corresponding Hubbard trees satisfy certain properties.

Definition 4.15 (Fatou and Julia vertices). *Let N_p be a Newton map and $\Delta_{N'}$ its Newton graph at level N'. A vertex $v \in \Delta_{N'}$ is called* a Fatou vertex *if it belongs to the Fatou set of N_p. Otherwise, it is called* a Julia vertex.

It is easy to see that a vertex $v \in \Delta_{N'}$ is a Fatou vertex if and only if it is eventually mapped by N_p into one of the superattracting fixed points of N_p, and all Julia vertices eventually land on ∞. Julia and Fatou vertices will be treated differently in the modification of U_k. Any edge in any Newton graph evidently joins a Fatou vertex with a Julia vertex. We first need to prove a fact about valency in the case when U_k and V_k share a common edge.

Lemma 4.16. *Let $[u_1, u_J]$ be an edge of $\Delta_{N'+m(k)}$ such that $[u_1, u_J] \subset \partial U_k \cap \partial V_k$, u_J is a Julia vertex, u_1 is a Fatou vertex, and k is fixed, $1 \leq k \leq M$. Then there is at most one other edge in ∂U_k having u_J as an endpoint.*

Proof. Suppose that there are two other edges $[u_2, u_J]$ and $[u_3, u_J]$ in ∂U_k where $u_2 \neq u_3$, Since $[u_1, u_J] \subset \partial V_k \cap \partial U_k$, we have from the construction that

$$[u_1, u_J] \subset \Delta_{N'} \cap \Delta_{N'+m(k)} = \Delta_{N'+m(k)}.$$

Observe that $\overline{\Delta_{m(k)} \setminus \Delta}$ is connected since $m(k) \geq N$, where the value of N comes from Theorem 3.15. Then, since $N' \geq N$, it follows from Corollary 3.11 that u_1, u_2, and u_3 all belong to the closure of the same component of $\overline{\Delta_{N'+m(k)} \setminus \Delta_{N'}}$. There is a simple path ℓ in this component (i.e. through $\overline{\Delta_{N'+m(k)} \cap V_k}$) which connects u_1 and u_2 (after possibly swapping the labels u_2 and u_3) so that the simply connected subdomain of V_k with boundary $[u_1, u_J], [u_2, u_J]$, and ℓ does not contain u_3 in its boundary. Thus U_k has more than one component: the subdomain just constructed, and a second distinct component containing u_3 in its boundary. This is a contradiction because U_k was constructed to be a single component. □

The modification contained in Lemma 4.19 makes frequent use of the following topological definition and proposition. The notation $U \subset\subset V$ means that U is compactly contained in V.

Definition 4.17 (The ε-neighborhood)**.** *Let K be a compact subset of $\widehat{\mathbb{C}}$.* The ε-neighborhood *of K is the set of points $x \in \widehat{\mathbb{C}}$ such that $d(x, K) < \varepsilon$, where d is the spherical metric in $\widehat{\mathbb{C}}$.*

Proposition 4.18. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous map and U, V subsets of $\widehat{\mathbb{C}}$ such that $U \subset\subset V$. Then $f(U) \subset\subset f(V)$.*

The proposition is an immediate consequence of the fact that f is a closed map, and the proof is omitted.

We finally turn to the discussion of how to alter the domains U_k produced in Lemma 4.14 to produce new domains $\widehat{U}_k$ on which we may define polynomial-like map without changing crucial dynamical properties.

Lemma 4.19 (Construction of renormalizations)**.** *For every proper map $F_k : U_k \to V_k$ constructed in Lemma 4.14, there exists a corresponding domain $\widehat{U}_k$ with $U_k \cap \widehat{U}_k \neq \emptyset$ so that the restriction $\widehat{F}_k := N_p^{m(k)}|_{\widehat{U}_k}$ is a polynomial-like map $\widehat{F}_k : \widehat{U}_k \to \widehat{V}_k$ (where $\widehat{V}_k := N_p^m(\widehat{U}_k)$) having the properties that* $\deg \widehat{F}_k = \deg F_k$ *and $K(\widehat{F}_k, \widehat{U}_k, \widehat{V}_k) = K(F_k, U_k, V_k)$.*

Proof. Fix k. If $\partial U_k \cap \partial V_k = \emptyset$, then let $\widehat{U}_k = U_k$ and $\widehat{V}_k = V_k$. However it might happen that $\partial U_k \cap \partial V_k \neq \emptyset$. This intersection may only consist of vertices and edges in the Newton graph $\Delta_{N'}$. We will modify U_k slightly near its boundary to produce a new set $\widehat{U}_k$ where $\widehat{U}_k \subset\subset \widehat{V}_k = N_p^m(\widehat{U}_k)$.

The modification is divided into three steps. The first step describes how to construct neighborhoods of Julia vertices and the second step does the construction for edges. The third step uses these neighborhoods to modify U_k to produce $\widehat{U}_k$. For brevity, we write m instead of $m(k)$.

Step 1–Neighborhood of Julia vertices: Since ∞ is a repelling fixed point of N_p^m, there exists $\varepsilon > 0$ and an open disk $\Omega(\infty)$ of radius ε centered at ∞ so that $\Omega(\infty) \subset\subset N_p^m(\Omega(\infty))$ and N_p^m is univalent on $\Omega(\infty)$.

For every prepole $J^1 \in N_p^{-m}(\infty)$, there is a connected component of $N_p^{-m}(\Omega(\infty))$ that contains J^1. Denote by $\Omega(J^1)$ a slightly larger domain compactly containing this connected component and lying in its ε-neighborhood. By Proposition 4.18,

$$\Omega(\infty) \subset\subset N_p^m(\Omega(J^1)). \tag{2}$$

Then, for every prepole $J^n \in N_p^{-nm}(\infty)$, $n > 1$, inductively construct a neighborhood $\Omega(J^n)$ of J^n so that

$$\Omega(N_p^m(J^n)) \subset\subset N_p^m(\Omega(J^n)), \quad n > 1. \tag{3}$$

Step 2–Neighborhood of edges: Let e be an edge of the channel diagram connecting a fixed critical point a of N_p of multiplicity $d-1 \geq 1$ to ∞. Let U be the immediate basin of a. The dynamics of N_p^m on U are conjugate to $z \mapsto z^{d^m}$ on the unit disk by the Böttcher coordinate. Denote by $\widetilde{e} := e \setminus \Omega(\infty)$ the part of the edge e that is not yet covered. Let $\Omega(\widetilde{e}) \subset U$ be the preimage of the disk of radius $1 - \varepsilon'$ centered at the origin under the Böttcher coordinate, where ε' is small enough so that $\Omega(\widetilde{e})$ contains $\widetilde{e}$, and $\partial\Omega(\widetilde{e})$ is contained in an ε neighborhood of ∂U. From the mapping properties of $z \mapsto z^{d^m}$, it is evident that

$$N_p^m(\Omega(\widetilde{e})) \subset\subset \Omega(\widetilde{e}).$$

Now we inductively construct neighborhoods for preimages of fixed edges under the map N_p^m. For every preimage E^1 of $\widetilde{e}$ under N_p^m, there is some connected component of $N_p^{-m}(\Omega(\widetilde{e}))$ that contains E^1. This connected component compactly contains a slightly smaller domain $\Omega(E^1)$ in the ε-neighborhood of E^1. Hence

$$N_p^m(\Omega(E^1)) \subset\subset \Omega(\widetilde{e}). \tag{4}$$

For every preimage E^n of $\widetilde{e}$ under N_p^{nm}, $n > 1$, use the same method to inductively construct neighborhoods $\Omega(E^n)$ within the ε-neighborhood of E^n so that

$$N_p^m(\Omega(E^n)) \subset\subset \Omega\left(N_p^m\left(E^n\right)\right), \quad n > 1. \tag{5}$$

Choose ε' from before sufficiently small so that the following holds: for every edge E^n that has a Julia vertex J^n as an endpoint, we have that $\partial\Omega\left(E^n\right)$ intersects $\partial\Omega(J^n)$ at precisely two different points.

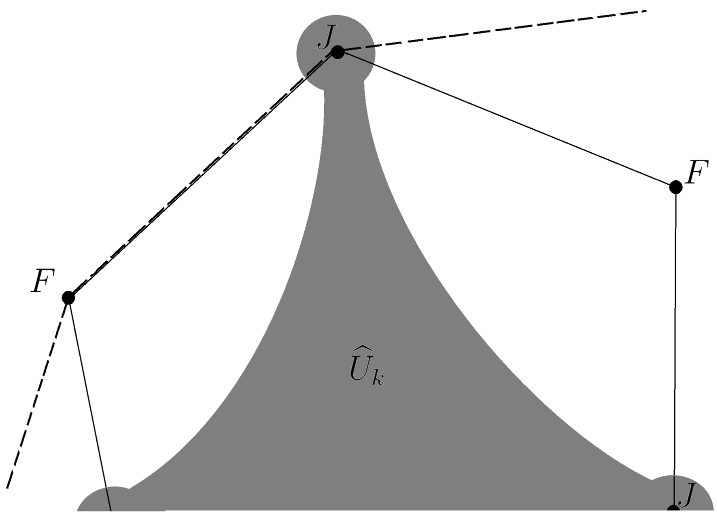

FIGURE 2. The boundaries of domains U_k and V_k are respectively represented by the solid and dashed polygonal arcs. In this example ∂U_k and ∂V_k intersect over a single edge. Fatou vertices are indicated by F, Julia vertices by J. The shaded region is the modified domain $\widehat{U}_k$ constructed in step 3.

Step 3–Construction of new domain and range: Let $\Omega(E)$ be the union of all neighborhoods around edges in ∂U_k as constructed in Step 2, and define $l(E) := \partial\Omega(E) \cap U_k$. For every Julia vertex $J^n \in \partial U_k$, denote by $l(J^n)$ the unique connected component of $\partial\Omega(J^k) \setminus l(E)$ that intersects $\widehat{\mathbb{C}} \setminus U_k$. Denote by $l(J)$ the union of $l(J^n)$ for all Julia vertices $J^n \in \partial U_k$. Let

$$l(U_k) = l(E) \cup l(J).$$

It follows that $l(U_k)$ is a simple closed curve with one complementary component in $\widehat{\mathbb{C}}$ that contains all Julia vertices in ∂U_k: call this component $\widehat{U}_k$. Letting $\widehat{V}_k = \widehat{F}_k(\widehat{U}_k)$ and $\widehat{F}_k = N_p^m|_{\widehat{U}_k}$, it follows from the compact containments in (2)-(5) that $\widehat{U}_k \subset\subset \widehat{V}_k$. Because N_p is postcritically finite, we may choose sufficiently small $\varepsilon > 0$ so that

$$P_{N_p} \cap \widehat{U}_k = P_{N_p} \cap U_k,$$

$$K(\widehat{F}_k, \widehat{U}_k, \widehat{V}_k) = K\left(F_k, U_k, V_k\right), \quad \text{and} \quad \deg(\widehat{F}_k) = \deg(F_k).$$

□

Note that for each polynomial-like map $\widehat{F}_k : \widehat{U}_k \to \widehat{V}_k$ constructed in Lemma 4.19 there is an extended Hubbard tree that we denote $H(U_k)$ whose cycle type is chosen to be minimal so that all postcritical points of N_p in $\widehat{U}_k$ are vertices.

Proposition 4.20. *Each extended Hubbard tree $H(U_k)$ is independent of the choice of domain $\widehat{U}_k$ and does not intersect the Newton graph. Any two distinct extended Hubbard trees $H(U_k)$ and $H(U_l)$ are disjoint.*

Proof. Independence of choice of domain is an immediate consequence of the construction and Theorem 4.13.

Though U_k was defined to be a complementary component of the Newton graph $\Delta_{n'}$ of some level n', the perturbed set $\widehat{U}_k$ intersects $\Delta_{n'}$ in small neighborhoods of Julia vertices in ∂U_k. By decreasing ε from the proof of Lemma 4.19, these neighborhoods can be made arbitrarily small, where the hypotheses of Theorem 4.13 are satisfied for each ε. It follows that the filled Julia set is unchanged for arbitrarily small ε. Thus $K(\widehat{F}_k, \widehat{U}_k, \widehat{V}_k)$ may not intersect any edges or Fatou vertices in $\Delta_{n'}$.

Suppose that $H(U_k)$ intersects a Julia vertex; clearly $H(U_k)$ may not be a degenerate Hubbard tree. Since $H(U_k)$ is forward invariant, and every Julia vertex maps to ∞, the tree $H(U_k)$ must also contain ∞. Since ∞ is a repelling fixed point and the channel diagram edges are invariant, $N_p(H(U_k))$ is in the same component of $\Delta_{n'}$ as $H(U_k)$. It follows that $N_p(H(U_k)) = H(U_k)$. Then N_p has some other fixed point in $H(U_k)$ distinct from ∞ (and by construction of $\widehat{U}_k$ distinct from any of the fixed critical points in the channel diagram). This contradicts the fact that all fixed points of N_p are in the channel diagram.

Finally, any two different extended Hubbard trees are clearly disjoint, as they are subsets of distinct complementary components of the Newton graph. □

5. Newton rays from Newton maps

We now construct Newton ray edges, which will connect the repelling fixed points of the extended Hubbard trees constructed in the previous chapter to the Newton graph. They are defined as subsets of *bubble rays*, which are chains of Fatou components of that have been used in the literature in several situations [YZ01, Roe98, Luo93].

The main results in this section are Lemma 5.6 which guarantees the existence of periodic rays landing on Hubbard trees, and Lemma 5.13 which guarantees that such rays can be found with minimal period.

Definition 5.1. *A* bubble *of N_p is a Fatou component in the basin of attraction of one of the fixed critical points of N_p. The* center *of a bubble B is the unique point of B which eventually maps to a fixed critical point under N_p.*

A bubble B is said to be *attached* or *adjacent* to another bubble B' if $\overline{B} \cap \overline{B'} \neq \emptyset$. A finite sequence of adjacent bubbles is called a *bubble chain.* The Newton map N_p has degree d, and recall that the collection of fixed critical points was denoted $a_1, a_2, ..., a_d$. Denote by A_i the immediate basin of a_i, $1 \leq i \leq d$. Let

$$\mathscr{A}^0 = A_1 \cup ... \cup A_d.$$

The *generation of B* is defined to be the minimal number $Gen(B)$ so that $N_p^{Gen(B)} = A_i$ for some i.

Definition 5.2 (Bubble rays)**.** *Let $B_0, B_1, \dots$ be an infinite sequence of distinct bubbles where B_0 is an immediate basin, and B_i is adjacent to B_{i-1} for all $i \geq 1$. Then the set*

$$\overline{\bigcup_{j \geq 0} B_j}$$

is called a bubble ray*, and*

$$\overline{\bigcup_{1 \leq j \leq m} B_j}$$

is called a finite bubble ray*. The bubble B_m is called the* end *of the finite ray, and the ray is said to have* generation *m.*

Subhyperbolicity of N_p implies that the diameter of B_j decays exponentially as j increases and so the tail of bubble ray $\mathcal{B} = \overline{\bigcup_{j \geq 0} B_j}$ converges to a unique point which we denote $t(\mathcal{B})$. We say that $\mathcal{B}$ *lands* at $t(\mathcal{B})$.

The notion of internal ray will now be defined for any bubble B. Recall that $N_p^{\text{Gen}(B)}(B) = A_i$ for some immediate basin A_i, $1 \leq i \leq d$. As mentioned in Section 3, each immediate basin A_i has a global Böttcher coordinate which is used to define internal ray angles in A_i. Lift ϕ_i to B to define internal rays in B as well (this lift could have finite degree greater than one).

Let Δ_N be the Newton graph of N_p that satisfies the conditions of Definition 6.1. For a bubble ray $\mathcal{B}$, there is at least one simple path connecting ∞ to $t(\mathcal{B})$ consisting of the closures of internal rays from the bubbles of $\mathcal{B}$. A choice of such path is denoted $\mathcal{R}^*(\mathcal{B})$.

Definition 5.3. *The closure of the unique component of* $\mathcal{R}^*(\mathcal{B}) \setminus \Delta_N$ *containing infinitely many internal rays is called* the Newton ray *associated with* $\mathcal{B}$ *and is denoted* $\mathcal{R}(\mathcal{B})$*.*

Observe that a Newton ray intersects Δ_N in precisely one point.

Definition 5.4. *A Newton ray* $\mathcal{R}(\mathcal{B})$ *is said to be* periodic *if there exists an integer* $m \geq 1$ *such that* $N_p^m(\mathcal{R}(\mathcal{B})) = \mathcal{R}(\mathcal{B}) \cup \mathcal{E}$*, where* $\mathcal{E} \subset \Delta_N$ *is a finite union of edges of* Δ_N*. The smallest such* m *is* the period *of* $\mathcal{R}(\mathcal{B})$*.*

See Figure 5 for a schematic example of two bubble rays, from which a period two Newton ray is extracted (indicated by light gray edges).

We now define terminology used in the proof of the crucial Lemma 5.6. The lemma asserts the existence of periodic rays which we use to connect Hubbard trees to the Newton graph.

Predecessor bubbles. For every bubble B, we assign a predecessor bubble $\mathscr{P}(B)$ so that on the large, predecessors are preserved by N_p (see Lemma 5.5).

The predecessor of an immediate basin A_i is declared to be itself, namely $\mathscr{P}(A_i) = A_i$. For other bubbles, the predecessor is assigned by means of an explicitly chosen (but non-canonical) maximal subtree T_i in the Newton graph.

Choose a maximal subtree $T_0 \subset \Delta_0 = \Delta$. Inductively define $T_i \subset \Delta_i$ to be a maximal subtree of $N_p^{-1}(T_{i-1}) \cap \Delta_i$ with the additional assumption that $T_i \supset T_{i-1}$. By construction, it is clear that $N_p(T_i) \subset T_{i-1}$.

Let v and v' with $v \neq v'$ be two Fatou vertices (recall Definition 4.15) that are adjacent to the same Julia vertex in T_i, and at most one of v and v' are fixed points of N_p. We say that v *precedes* v' in T_i if v lies in the same complementary component of $T_i \setminus \{v'\}$ as the vertex at ∞.

We now define bubble predecessors in terms of T_i. Let B be a bubble that is not a fixed immediate basin, and let i be large enough so that T_i contains the center of B (the existence of such an i is a consequence of Theorem 3.10). Then the *predecessor bubble* $\mathscr{P}(B)$ is the bubble whose center is the predecessor of the center of B in T_i. Since the construction of the subtree T_i was non-canonical, the definition of predecessor bubbles is also non-canonical.

The bad set. Fix i large enough so that all poles and eventually fixed critical points of N_p are contained in T_{i-1}. We show that N_p preserves predecessors defined by T_i, except possibly when both vertices are chosen from a finite set of "bad" vertices that is independent of i. Let $\mathscr{S}$ denote the set of critical points and poles of N_p in T_i. The set V_{bad} is defined to be the set of Fatou vertices in T_i satisfying the following properties:

- the spanning tree of V_{bad} in T_i contains $N_p^{-1}(\mathscr{S})$
- V_{bad} is closed under predecessors (i.e. if $v \in V_{bad}$, then the predecessor of v is in V_{bad}).
- V_{bad} is minimal.

The set V_{bad} is evidently finite.

Lemma 5.5. *For all bubbles B that do not intersect V_{bad}, the following holds:*

$$N_p(\mathscr{P}(B)) = \mathscr{P}(N_p(B)).$$

Proof. Let v, v' be two vertices so that v precedes v', where v' is not in V_{bad}. There is an oriented geodesic γ connecting v to an end of T_i that passes through v' (observe that it can pass through no critical points since these are in V_{bad}). Then $N_p(\gamma)$ is an oriented geodesic in $N_p(T_i)$ which passes sequentially through $N_p(v)$, $N_p(v')$, and an end of $N_p(T_i)$. Since $N_p(\gamma)$ does not pass through the vertex at ∞, we see that $N_p(v)$ is the predecessor of $N_p(v')$. □

Lemma 5.6. *Let ω be a repelling periodic point of period $m > 1$ of N_p. Then there exists a periodic Newton ray $\mathcal{R}$ that lands at ω and has period given by an integer multiple of m.*

Proof. There exists some neighborhood Y of ω so that N_p^{-m} has some branch h with $h(Y) \subset Y$. We may further assume without loss of generality that Y intersects the minimal possible number of bubbles with centers in Δ_N. Evidently ω is an attracting fixed point of h, and thus there is some fundamental annulus $A \subset Y$ for the dynamics of h. Let B_0 be some bubble contained in A. The sequence of bubbles $B_n = h^n(B_0)$ evidently accumulate at ω.

For any bubble B, the associated bubble ray is given by

$$\widehat{B} = \bigcup_{k=0}^{\infty} \mathscr{P}^k(B).$$

It is evident that $\widehat{B}$ is a finite bubble ray because $\mathscr{P}^k(B)$ is eventually an immediate basin. Each $\widehat{B_n}$ is a sequence of bubbles which start at B_n and eventually leave Y, never to return.

To prove the lemma, it suffices to show that there is some $i > j$ so that $\widehat{B_i}$ and $\widehat{B_j}$ have a common bubble in Y for the following reason. Let B_i^j be the minimal sequence of adjacent bubbles in $\widehat{B_i} \cup \widehat{B_j}$ whose first bubble is B_i and whose last bubble is B_j. Then the following is a periodic Newton ray that lands at ω:

$$\mathcal{R}\big(N_p^{m\cdot\mathrm{Gen}(B_j)}(\bigcup_{k=1}^{\infty} h^k(B_i^j))\big).$$

We now show that there is some $i > j$ so that $\widehat{B_i}$ and $\widehat{B_j}$ have a common bubble in Y. Observe that $N_p^{Gen(B_0)}(B_0)$ is an immediate basin. We call iterated preimages of this immediate basin under N_p^m the *marked bubbles.* Note that the sequence of marked bubbles in each $\widehat{B_i}$ has the following properties (as $\widehat{B_i}$ is traversed from B_i to the Newton graph):

- The next marked bubble after a given marked bubble has lower generation, and there are at most M unmarked bubbles between the two, where M is the maximum over the length of any chain of predecessors connecting ∞ to a pole of N_p^m.

- Any bubble B between a marked bubble of generation k and a marked bubble of generation $k+i$ for some $i \geq 1$ has generation $k \leq \text{Gen}(B) \leq k+i+N$.

By the subhyperbolicity of N_p, there must be some n so that for $k \geq n$, $\widehat{B_k}$ has a sequence of $2M+2$ distinct marked bubbles so that all bubbles between the first and last marked bubbles are completely contained in the annulus A. The generations of *all* bubbles in $\widehat{B_k}$ is bounded by $\text{Gen}(B_0)+m \cdot k+N$. Let $\max_\infty(\widehat{B_i})$ denote the maximal generation of the bubbles of $\widehat{B_i}$ lying in the complementary component of the annulus containing ∞. Since every bubble chain of length $2M+2$ must contain at least two marked bubbles, it is evident that for $i > n$, $\max_\infty(\widehat{B_i}) < \max_\infty(\widehat{B_n}) + n \cdot m - i$. But choosing i large enough, this contradicts the fact that bubble generations are non-negative. □

Remark 5.7. For any Newton ray $\mathcal{R}$ with landing point $t(\mathcal{R})$,

$$(\mathcal{R} \setminus \{t(\mathcal{R})\}) \cap H(U_k) = \emptyset.$$

This is because any point $x \in \mathcal{R} \setminus \{t(\mathcal{R})\}$ is eventually mapped onto Δ_N by N_p, while the orbit of $H(U_k)$ under N_p is disjoint from Δ_N.

In Section 4, polynomial-like maps $\widehat{F}_k : \widehat{U}_k \to \widehat{V}_k$ of periods $m(k)$ were constructed, $1 \leq k \leq M$. We fix k for the rest of the section, and let ω be a repelling fixed point of $\widehat{F}_k$. We do not need to order rays that land at degenerate Hubbard trees, so we assume that the extended Hubbard tree $H(U_k)$ is not degenerate. Arbitrarily choose an edge in $H(U_k)$ with ω as an endpoint. Denote this edge by E_ω.

We spend the rest of this section showing how to produce a "rightmost" Newton ray in order to prove Lemma 5.13 which asserts the existence of a periodic Newton ray of minimal possible period.

Fix the orientation of $\mathbb{S}^2$ to be the counterclockwise orientation for the rest of this paper.

Definition 5.8 (Newton ray order)**.** *Let $\mathcal{R}'$, $\mathcal{R}''$ be Newton rays landing at ω and let E_w be an edge in $H(U_k)$ with endpoint ω. The Newton rays are said to* not cross-intersect *if they satisfy the following property: if l is a curve disjoint from $\mathcal{R}'$, $\mathcal{R}''$ and connecting the endpoints of $\mathcal{R}''$ and E_w different from ω, then $\mathcal{R}'$ intersects only one complementary component of $\widehat{\mathbb{C}} \setminus (E_\omega \cup l \cup \mathcal{R}'')$. Assume that $\mathcal{R}'$ and $\mathcal{R}''$ don't cross-intersect. Let Y be a neighborhood of ω such that for some branch $h = N_p^{-m}$, we have $h(Y) \subset Y$. We say that $\mathcal{R}' \succeq \mathcal{R}''$ if for any such neighborhood Y, the cyclic order around ω in Y is $\mathcal{R}'$, $\mathcal{R}''$, E_w.*

Remark 5.9. Note that for any other such neighborhood $Y' \subset Y$, the cyclic order of $\mathcal{R}''$, $\mathcal{R}'$, E_w in Y' is the same as in Y. Hence the relation $\succeq$ is well defined and doesn't depend on the choice of the neighborhood Y.

Lemma 5.10. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be periodic (possibly cross-intersecting) Newton rays that land at a repelling fixed point ω of the polynomial-like map $\widehat{F}_k$. Then there is a Newton ray $\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2)$ that satisfies the following propertes:*

- *$\mathcal{R}$ lands at ω.*
- *$\mathcal{R}$ doesn't cross-intersect either $\mathcal{R}_1$ or $\mathcal{R}_2$.*
- *$\mathcal{R} \succeq \mathcal{R}_1$, $\mathcal{R} \succeq \mathcal{R}_2$.*

Remark 5.11. Such a ray $RE(\mathcal{R}_1, \mathcal{R}_2)$ is said to be *the right envelope* of Newton rays $\mathcal{R}_1, \mathcal{R}_2$ (see Figure 3).

Proof. Let Y be a neighborhood of ω such that for some branch $h = N_p^{-m}$, $h(Y) \subset Y$, $E_w \cap Y \subsetneqq E_w$ and $\partial Y \cap \mathcal{R}_1 = \{v_1\}$, $\quad \partial Y \cap \mathcal{R}_2 = \{v_2\}$, where v_1, v_2 are iterated preimages of vertices of Δ_N under N_p. Let Y_1, Y_2 be the connected components of $Y \setminus (\mathcal{R}_1 \cup \mathcal{R}_2 \cup E_w)$ so that $E_w \cap Y \subset \partial Y_1 \cap \partial Y_2$ and the cyclic order around ω is Y_2, E_w, Y_1. We define what will be called the right envelope of $\mathcal{R}_1$ and $\mathcal{R}_2$ in Y by

$$RE_Y(\mathcal{R}_1, \mathcal{R}_2) := \partial Y_1 \setminus (E_w \cup \partial Y).$$

It follows from the construction that

$$RE_Y(\mathcal{R}_1, \mathcal{R}_2) \subset (\mathcal{R}_1 \cup \mathcal{R}_2) \cap Y.$$

Let i be the smallest integer $i > 0$ such that $N_p^{im}(RE_Y(\mathcal{R}_1, \mathcal{R}_2)) \cap \Delta_N \neq \emptyset$. Define

$$\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2) := \overline{N_p^{im}(RE_Y(\mathcal{R}_1, \mathcal{R}_2)) \setminus \Delta_N}.$$

Note that if two Newton rays intersect, then the bubble rays associated to them must have common bubbles containing the intersection points or edges over which the Newton rays intersect. Vice versa, every intersection point or a common edge of two Newton rays is contained in a bubble that is a common part of bubble rays associated to both of the Newton rays. Since the finiteness condition on generations of bubbles containing the edges of Newton rays $\mathcal{R}_1$ and $\mathcal{R}_2$ is satisfied, it also holds for $\mathcal{R}$. Thus $\mathcal{R}$ is a Newton ray that lands at ω; furthermore

$$\mathcal{R} \succeq \mathcal{R}_1, \mathcal{R} \succeq \mathcal{R}_2 \quad \text{and} \quad \mathcal{R} \subset \mathcal{R}_1 \cup \mathcal{R}_2.$$

□

Remark 5.12. Note that the construction of the Newton ray $RE(\mathcal{R}_1, \mathcal{R}_2)$ doesn't depend on the choice of Y. The right envelope $RE(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n)$ of finitely many Newton rays $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ is defined analogously.

Lemma 5.13. *For any repelling fixed point ω of a polynomial-like map $\widehat{F}_k : \widehat{U}_k \to \widehat{V}_k$ of period m and degree greater than one, there exists a Newton ray of period $m \cdot \ell$ that lands at ω, where ℓ is the period of the external rays landing at ω under $\widehat{F}_k^m$.*

Proof. It follows from Lemma 5.6 that there exists a positive integer r and a Newton ray $\mathcal{R}_1$ of period mr that lands at ω. Let $\mathcal{R}_i = N_p^{(i-1)\cdot m}(\mathcal{R}_1)$ for $1 \leq i \leq r$ and recall that the right envelope

$$\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_r)$$

is a periodic Newton ray that lands at ω. Denote by Y the neighborhood of ω such that for some branch $h = N_p^{-m}$, $h(Y) \subset Y$ and let Y_1 be the connected component of

$$Y \setminus \bigcup_{i=1}^{r} \mathcal{R}_i$$

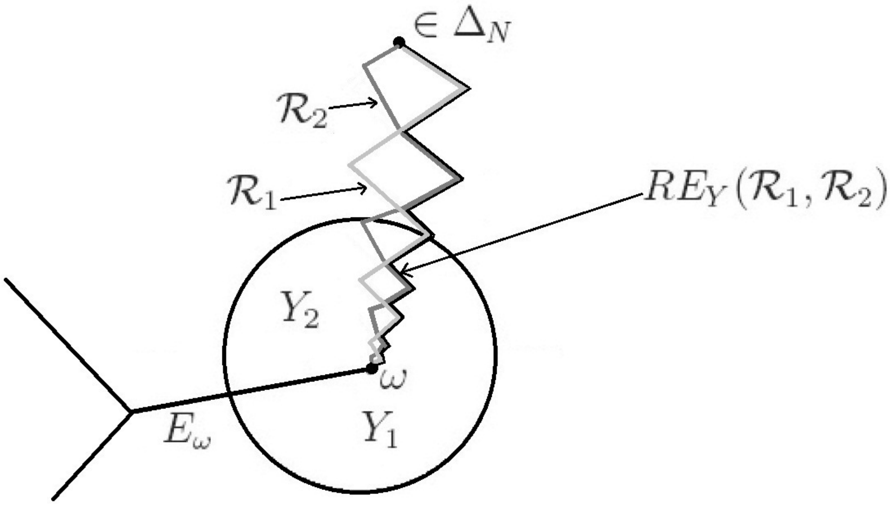

FIGURE 3. Two rays $\mathcal{R}_1$ and $\mathcal{R}_2$ that cross intersect and land at the point ω are indicated by light and medium gray. The disk represents the set Y, and the right envelope $RE_Y(\mathcal{R}_1, \mathcal{R}_2)$ is indicated by the jagged black line.

such that

$$\mathcal{R} \cap \partial Y_1 \neq \emptyset \quad \text{and} \quad E_\omega \cap \partial Y_1 \neq \emptyset.$$

Since the map N_p is orientation preserving,

$$N_p^{m\cdot\ell}(Y_1) \cap Y_1 = Y_1,$$

and because the $\mathcal{R}_i$ form a cycle, $N_p^{m\cdot\ell}(\mathcal{R}) = \mathcal{R} \cup \mathcal{E}$, where $\mathcal{E}$ is a union of edges of Δ_N. Therefore $RE(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_r)$ is a Newton ray of period $m \cdot \ell$. □

6. Extended Newton graphs from Newton maps

6.1. **Construction.** In Section 3, the Newton graph of arbitrary level was constructed for a given Newton map N_p. Next, extended Hubbard trees were constructed in Section 4 to give a combinatorial description of the periodic free postcritical points. Finally, periodic and preperiodic Newton rays were constructed in Section 5 to connect the Hubbard trees to the Newton graph.

Here we specify the level of the Newton graph that will be used in the construction of the extended Newton graph.

Definition 6.1 (Newton graph of a postcritically finite Newton map)**.** *For a postcritically finite Newton map N_p, let N be the minimal integer such that*

- *no two different extended Hubbard trees $H(U_k)$, $1 \leq k \leq M$ lie in the same complementary component of Δ_N.*
- *every critical point that eventually lands on the channel diagram Δ is contained in Δ_N.*

The graph Δ_N is called the Newton graph *of N_p.*

The proof of the following theorem uses these objects to construct a connected finite forward-invariant graph Δ_N^* containing the postcritical set. This graph will then be defined to be the extended Newton graph of N_p.

Theorem 6.2. *For a given postcritically finite Newton map N_p, let Δ_N be the Newton graph of N_p. There exists a finite connected graph Δ_N^* that contains Δ_N, is invariant under N_p and contains the critical and postcritical set of N_p. Furthermore, every edge of Δ_N^* is eventually mapped by N_p either into Δ_N, into an extended Hubbard tree, or onto a periodic Newton ray union edges from Δ_N.*

Proof. The Newton graph Δ_N captures the behavior of postcritical points of N_p which eventually map into the channel diagram Δ. We now deal with the postcritical points of N_p which are not eventually fixed.

Postcritical points contained in periodic Hubbard trees: Let w_k be some periodic postcritical point. It follows from Lemma 4.19 that there exist domains $\widehat{U}_k \subset \widehat{V}_k$, such that $w_k \in \widehat{U}_k$ and

$$\widehat{F}_k = N_p^{m(k)} : \widehat{U}_k \to \widehat{V}_k$$

is a polynomial-like map. There is associated to $\widehat{F}_k$ an extended Hubbard tree $H(U_k)$ containing w_k (which could possibly be degenerate). Furthermore, all postcritical points in $\widehat{U}_k$, including preperiodic ones, are contained in $H(U_k)$.

By Lemma 5.13 there is a period $m(k) \cdot \ell(k)$ Newton ray $\gamma(U_k)$ that lands at a repelling fixed point of $\widehat{F}_k$ contained in $H(U_k)$. Denote by $\tilde{\gamma}(U_k)$ all rays in $N_p^{-1}(N_p(\gamma(U_k)))$ that land on $H(U_k)$. We define

$$\Upsilon(w_k) = \Big[\bigcup_{i=}^{m(k)-1} N_p^i(H(U_k)) \cup \bigcup_{i=0}^{m(k)-1} N_p^i(\tilde{\gamma}(U_k))\Big] \setminus \Delta_N.$$

Then $\Upsilon(w_k) \cup \Delta_N$ is a connected forward invariant graph that is a union of Newton ray, Newton graph, and Hubbard tree edges. All edges in the graph are disjoint, except possibly at their endpoints.

Pre-periodic postcritical points mapping into a periodic Hubbard tree: Now suppose that some critical point w_j is not contained in the forward orbit of any other critical point, and $w_j \notin \widehat{U}_k$ for any k. Let $r > 0$ be the minimal choice of integer so that $N_p^r(w_j)$ is contained in a periodic Hubbard tree which we denote $H(U_r)$. The periodic ray connecting $H(U_r)$ to the Newton graph is denoted $\gamma(U_r)$.

Define $w_j^{(i)} := N_p^i(w_j)$. Denote by $H^i(U_r)$ the connected component of $N_p^{i-r}(H(U_r))$ that contains $w_j^{(i)}$. For $0 \le i < r$, we describe inductively how to construct the preperiodic Newton rays $\gamma^i(U_r)$ that connect $H^i(U_r)$ to the Newton graph so that

$$N_p(\gamma^i(U_r)) \subset \gamma^{i+1}(U_r) \cup \Delta_N. \tag{6.1}$$

For $i = r-1$ let $\tilde{\gamma}^i(U_r)$ be all ray components of $N_p^{-1}(\gamma(U_r))$ that land at $H^i(U_r)$ subject to the following modification. If any ray in $\tilde{\gamma}^i(U_r)$ does not intersect Δ_N, extend the ray by a simple path in $\Delta_{2N} \setminus \Delta_N$ to produce a ray connecting the endpoint of $\tilde{\gamma}^i(U_r)$ to Δ_N; all such extensions can and must be chosen to be disjoint.

Proceed similarly to construct such a $\tilde{\gamma}^i(U_r)$ for all $0 \leq i < r$, and define

$$\Upsilon(w_j^{(i)}) = H^i(U_r) \cup \tilde{\gamma}^i(U_r).$$

Construction of the graph: Choose a single periodic postcritical point in the orbit of each periodic Hubbard tree. Let P be the union of these points together with all postcritical points not contained in periodic Hubbard trees. The graph satisfying the conclusion of the theorem is

$$\Delta_N^* = \Delta_N \cup \bigcup_{w \in P} \Upsilon(w) \tag{6.2}$$

As constructed, an extended Newton graph Δ_N^* with Newton ray edges will have infinitely many vertices since each Newton ray is composed of a sequence of infinitely many preimages of edges. We now alter the edge set and vertex set of Δ_N^* to produce a finite graph without changing the topology of Δ_N^* as a subset of $\mathbb{S}^2$. Each (periodic and pre-periodic) Newton ray is taken as a single edge, thereby eliminating all of the vertices in the Newton ray except its endpoints. For convenience, we still denote this new finite graph by Δ_N^*. Thus the vertices of Δ_N^* are the vertices of Δ_N, the vertices of the Hubbard trees (which are chosen to include repelling fixed points of the polynomial-like restrictions and postcritical points of N_p in the filled Julia sets), and points in the Hubbard tree preimages which map to these vertices. This graph is finite, connected, forward invariant under N_p and contains the whole postcritical set of N_p. Moreover, every edge of Δ_N^* is evidently mapped by N_p in the required way. □

Definition 6.3 (Extended Newton graph)**.** *We call the pair* $(\Delta_N^*, N_p|_{\Delta_N^*})$ *from Equation 6.2 an* extended Newton graph *associated to* N_p*.*

The following proposition asserts that the extended Newton graph assigned to a Newton map is unique on the Newton graph and Hubbard tree edges (though of course uniqueness is not expected for the Newton rays). It is a consequence of Proposition 4.20 and the construction.

Proposition 6.4. *Let* $(\Delta_{N,1}^*, N_p)$ *and* $(\Delta_{N,2}^*, N_p)$ *be two extended Newton graphs constructed for* N_p*, and denote by* $\Delta_{N,1}^-$ *and* $\Delta_{N,2}^-$ *the respective graphs with all Newton ray edges removed. Then* $\Delta_{N,1}^- = \Delta_{N,2}^-$ *and* $N_p|_{\Delta_{N,1}^-} = N_p|_{\Delta_{N,2}^-}$*.*

6.2. **Example.** There is a postcritically finite Newton map N_p associated to a monic polynomial p whose roots are given approximately by

$$a_1 = 1, a_2 = -1, a_3 = -0.0094672882 + .3728674604i,$$

$$a_4 = -0.0094672882 - .3728674604i$$

that satisfies the following: the roots of p are simple critical points of N_p, and N_p has two additional real critical points at $z \approx 0.3740835220, -.3835508102$ lying in a two cycle and a four cycle respectively. Figure 4 displays the dynamical plane of N_p.

Since the polynomial p has real coefficients, it is evident that N_p must have a $z \mapsto \overline{z}$ symmetry.

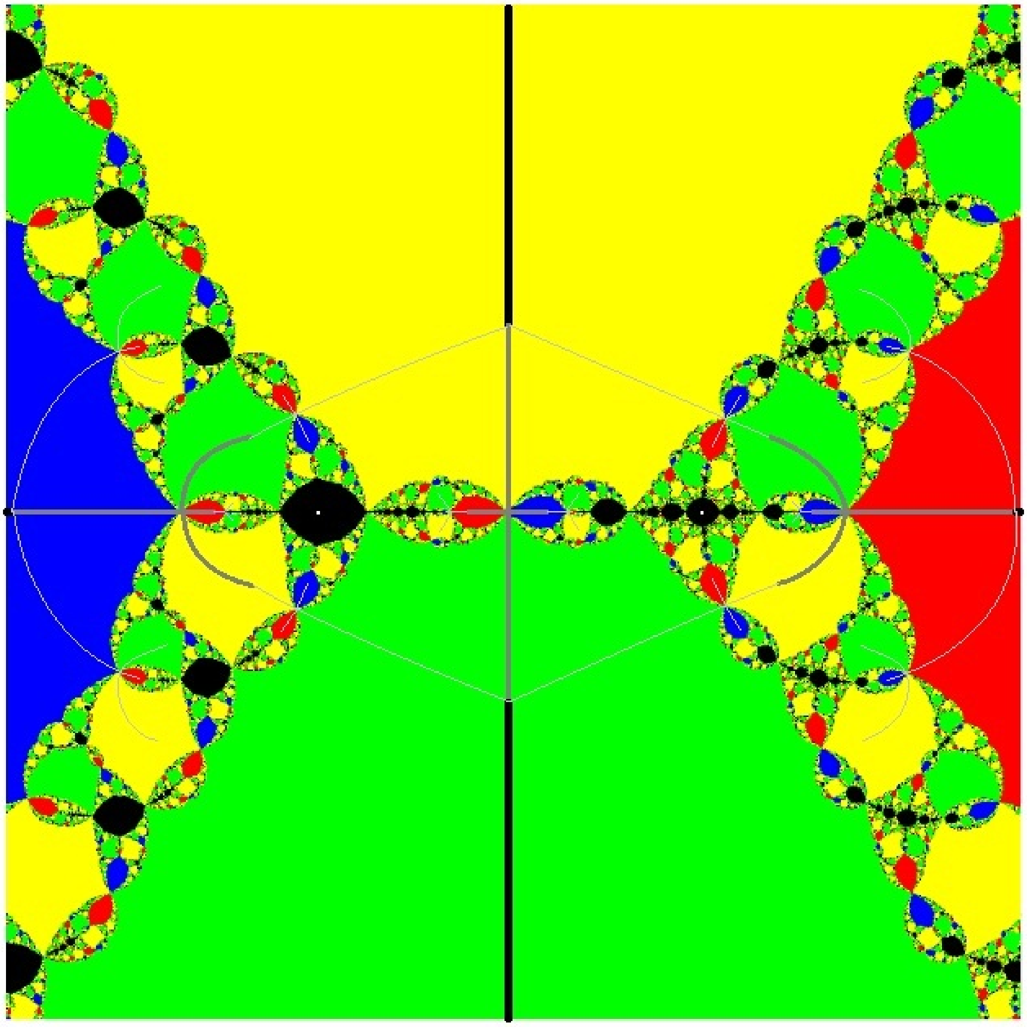

FIGURE 4. Part of the dynamical plane for N_p from Section 6.2. The largest colored regions are the immediate basins of the roots. The channel diagram is indicated by black lines. The edges in $\Delta_1 \setminus \Delta$ are indicated by thinner lines with a lighter shade, and the edges of $\Delta_2 \setminus \Delta_1$ are indicated by even thinner and lighter lines. The union of all three kinds of lines indicates the Newton graph. Two white dots in the black filled Julia set indicate free critical points.

The Newton graph of N_p is taken to be the Newton graph of level two (see Definition 6.1).

Renormalization (in the sense of Section 4) at either of the free simple critical points yields a degree four polynomial-like map. The corresponding filled Julia sets each contain a simple critical point of N_p and are mapped 2:1 onto each other by N_p. The renormalization has three fixed points in the

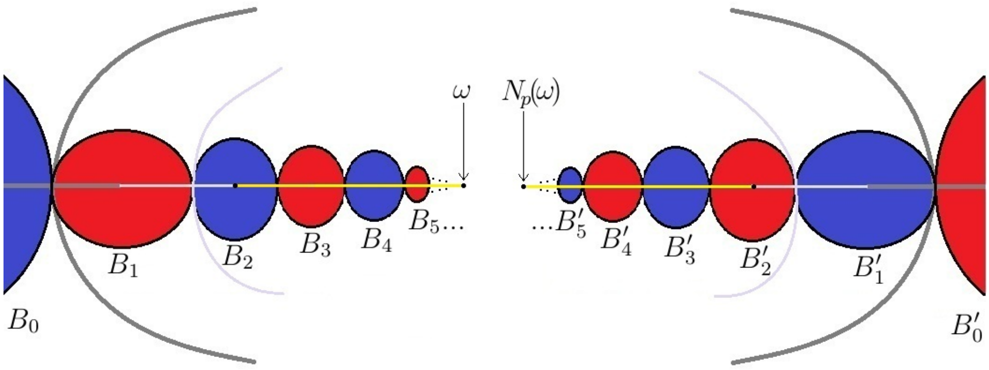

FIGURE 5. A topological model of the bubble and Newton rays for Example 1 landing at $\omega = N_p^2(\omega)$ and $N_p(\omega)$. The bubbles B_0 and B_0' correspond to the immediate basins of roots, and the dark grey edges correspond to edges in the Newton graph. The light grey edges are the Newton rays, which have period 2. For $i \neq 0$, $N_p(B_i) = B_{i-1}'$ and $N_p(B_i') = B_{i-1}$

left-hand filled Julia set (which contains the critical point $-.3835508102$). The fixed points are given approximately by:

- $-.3835508102$ indicated by a white dot in the figure.
- $\omega \approx -0.5531911255$, the left most endpoint of the filled Julia set.
- the unique point in the filled Julia set that lies in the closure of the immediate basins of the Newton map which contain non-real roots.

We now connect ω and $N_p(\omega)$ to the Newton graph by periodic Newton rays (see Figure 5). Denote by B_0 the immediate basin of the negative real root of p, and denote by B_0' the immediate basin of the positive root. Let B_1 be the unique preimage of B_0' that is not an immediate basin and is adjacent to B_0, and define B_1' similarly. Inductively define B_i to be the unique preimage of B_{i-1}' that is adjacent to B_{i-1}, and define B_i' similarly. Note that for $i \neq 0$, we have $N_p(B_i) = B_{i-1}'$ and $N_p(B_i') = B_{i-1}$. Furthermore, the B_i accumulate on ω, and the B_i' accumulate on $N_p(\omega)$. Let $\mathcal{B}, \mathcal{B}'$ denote the bubble ray composed of the B_i, B_i' respectively. Then note that the corresponding Newton rays $\mathcal{R}(\mathcal{B}), \mathcal{R}(\mathcal{B}')$ form a two-cycle that connect the extended Hubbard trees of the filled Julia sets to the Newton graph.

The extended Newton graph Δ_N^* is now defined as follows. The vertices are the vertices of the Newton graph and the vertices of the two extended Hubbard trees containing the free critical points. The edges are the edges of the Newton graph, the edges of the two extended Hubbard trees, and the two periodic Newton rays just constructed together with all preimages of the Newton rays that land on the two extended Hubbard trees.

7. Abstract extended Newton graphs

In this section we define the abstract axiomatizations that describe the Newton graph together with its extension by Hubbard trees and Newton rays, and then we show that every postcritically finite Newton map indeed has extended Newton graphs that satisfy these axioms, as claimed in Theorem 1.2. The converse that every abstract extended Newton graph is indeed realized by a postcritically finite map is true; this will be proved in [LMS].

Abstract Newton rays must first be defined. Let Γ be a finite connected graph embedded in $\mathbb{S}^2$ and $f : \Gamma \to \Gamma$ a weak graph map so that after promoting it to a graph map in the sense of 3.9, it can be extended to a branched cover $\overline{f} : \mathbb{S}^2 \to \mathbb{S}^2$.

Definition 7.1. A periodic abstract Newton ray $\mathcal{R}$ with respect to (Γ, f) *is an arc in* $\mathbb{S}^2$ *which satisfies the following:*

- *one endpoint of* R *is a vertex* $i(R) \in \Gamma$ *(the "initial endpoint"), and the other endpoint is a vertex* $t(R) \in \mathbb{S}^2 \setminus \Gamma$ *(the "terminal endpoint").*
- $\mathcal{R} \cap \Gamma = \{i(\mathcal{R})\}$.
- *there is a minimal positive integer* m *so that* $\overline{f}^m(\mathcal{R}) = \mathcal{R} \cup \mathcal{E}$*, where* $\mathcal{E}$ *is a finite union of edges from* Γ.
- *for every* $x \in \mathcal{R} \setminus t(\mathcal{R})$*, there exists an integer* $k \geq 0$ *such that* $\overline{f}^k(x) \in \Gamma$.

We say that the integer m *is* the period *of* $\mathcal{R}$*, and that* $\mathcal{R}$ lands *at* $t(\mathcal{R})$.

Definition 7.2. *A* preperiodic abstract Newton ray $\mathcal{R}'$ with respect to (Γ, f) *is an arc in* $\mathbb{S}^2$ *which satisfies the following:*

- *one endpoint of* R' *is a vertex* $i(R') \in \Gamma$*, and the other endpoint is* $t(R') \in \mathbb{S}^2 \setminus \Gamma$.
- $\mathcal{R}' \cap \Gamma = \{i(\mathcal{R}')\}$.
- *there is a minimal integer* $l > 0$ *such that* $\overline{f}^l(\mathcal{R}') = \mathcal{R} \cup \mathcal{E}$*, where* $\mathcal{E}$ *is a finite union of edges of* Γ *and* $\mathcal{R}$ *is a periodic abstract Newton ray with respect to* (Γ, f).
- $\mathcal{R}'$ *is not a periodic abstract Newton ray with respect to* (Γ, f).

We say that the integer l *is* the preperiod *of* $\mathcal{R}'$*, and that* $\mathcal{R}'$ lands *at* $t(\mathcal{R}')$.

Now we will introduce the concept of an abstract extended Newton graph. In [LMS], this graph will be shown to carry enough information to characterize postcritically finite Newton maps.

Definition 7.3 (Abstract extended Newton graph)**.** *Let* $\Sigma \subset \mathbb{S}^2$ *be a finite connected graph, and let* $f : \Sigma \to \Sigma$ *be a weak graph map. A pair* (Σ, f) *is called an* abstract extended Newton graph *if the following are satisfied:*

(1) (Edge Types) Any two different edges in Σ *may only intersect at vertices of* Σ*. Every edge must be one of the following three types (defined respectively in items (2),(3-4), and (6-7) below):*
   - *Type N: An edge in the abstract Newton graph* Γ
   - *Type H: An edge in a periodic or pre-periodic abstract Hubbard tree*
   - *Type R: A periodic or pre-periodic abstract Newton ray with respect to* (Γ, f).

(2) *(Abstract Newton graph) There exists a positive integer N and an abstract Newton graph Γ at level N so that $\Gamma \subseteq \Sigma$. Furthermore N is minimal so that condition (5) holds.*
(3) *(Periodic Hubbard trees) There is a finite collection of (possibly degenerate) minimal abstract extended Hubbard trees $H_i \subset \Sigma$ which are disjoint from Γ, and for each H_i there is a minimal positive integer $m_i \geq 2$ called the* period of the tree *such that $f^{m_i}(H_i) = H_i$.*
(4) *(Preperiodic trees) There is a finite collection of possibly degenerate trees $H_i' \subset \Sigma$ of preperiod ℓ_i, i.e. there is a minimal positive integer ℓ_i so that $f^{\ell_i}(H_i')$ is a periodic Hubbard tree (H_i' is not necessarily a Hubbard tree). Furthermore for each i, the tree H_i' contains a critical or postcritical point.*
(5) *(Trees separated) Any two different periodic or pre-periodic Hubbard trees lie in different complementary components of Γ.*
(6) *(Periodic Newton rays) For every periodic abstract extended Hubbard tree H_i of period m_i, there is exactly one periodic abstract Newton ray $\mathcal{R}_i$ connecting H_i to Γ. The ray lands at a repelling fixed point $\omega_i \in H_i$ and has period $m_i \cdot r_i$ where r_i is the period of any external ray landing at the corresponding fixed point of the polynomial realizing H_i.*
(7) *(Preperiodic Newton rays) For every preperiodic tree H_i', there exists at least one preperiodic abstract Newton ray in Σ connecting H_i' to Γ. A preperiodic ray landing at a periodic Hubbard tree must have preperiod 1.*
(8) *(Unique extendability) The conditions of Proposition 3.7 are met; thus f has a regular extension $\overline{f}$ which is unique up to Thurston equivalence (after upgrading f to a graph map following Remark 3.9).*
(9) *(Topological admissibility) The total number of critical points of f in Σ counted with multiplicity is $2d_\Gamma - 2$, where d_Γ is the degree of the abstract channel diagram $\Delta \subset \Gamma$.*

Remark 7.4 (Vertices and mapping properties of the graph). The set of vertices of the extended Newton graph is taken to be the (finite) collection of all Hubbard tree and Newton graph vertices.

Now we are going to give the proof of our main theorem which states that an extended Newton graph of a postcritically finite Newton map is indeed an abstract extended Newton graph.

Proof of Theorem 1.2. For a given Newton map N_p consider the extended Newton graph Δ_N^* from Definition 6.3. We show that (Δ_N^*, N_p) is an abstract extended Newton graph by verifying conditions (1) – (9) of Definition 7.3.

(1) By construction, every edge of Δ_N^* is either type N, H, or R. We show that the edges of each type may intersect only over vertices. By Proposition 4.20, type H edges may not intersect type N edges, and by construction the intersections with other type H edges may only be over vertices. It follows from Remark 5.7 that the interiors of edges of type H are also disjoint from edges of type R. Any two type R edges are contained in distinct complementary component of the Newton graph. Finally, by Definition 5.3,

the edges of type N and edges of type R can only intersect at vertices of Δ_N.

(2) Let Δ_N be the Newton graph of N_p as in Definition 6.1. Then (Δ_N, N_p) satisfies the properties of an abstract Newton graph by Theorem 3.15. Minimality is immediate.

(3) The extended Hubbard trees $H(U_k)$ constructed in Theorem 6.2 for periodic postcritical points z_k of N_p are periodic and satisfy the properties of abstract extended Hubbard trees (Theorem 4.3). Proposition 4.20 states that there is no common vertex with the Newton graph.

(4) Also by construction of Δ_N^*, the trees associated to preperiodic postcritical points z_k of N_p are preimages of periodic Hubbard trees under iterates of N_p. Since periodic Hubbard trees may not intersect the Newton graph, the preimage trees may have no common vertex with the Newton graph.

(5) The existence of such a level of the Newton graph so that the trees are separated is a consequence of the construction of the domains of renormalization in Lemmas 4.14 and 4.19.

(6), (7) Every periodic Newton ray (see Definition 5.3) is easily shown to be a periodic abstract Newton ray, and the corresponding statement holds for preperiodic rays. The rest of the properties follow immediately from the construction.

(8) Theorem 6.2 states that the whole critical and postcritical set of N_p is contained in the extended Newton graph Δ_N^*. Furthermore, if v is a critical point of N_p with $N_p(v) = w$, the valence of the graph at v is equal to the product of $\deg_v(N_p)$ and the valence of the graph at w. Thus the hypothesis of Proposition 3.7 holds.

(9) Since the degree of N_p equals the degree of its channel diagram, the conclusion follows from the Riemann-Hurwitz formula. □

8. Conclusion

We have shown how to extract a graph from any postcritically finite Newton map that satisfies the defining properties of an abstract extended Newton graph. In [LMS], it will be shown that every abstract extended Newton graph is realized by a postcritically finite Newton map. An equivalence relation will be placed on the set of all abstract extended Newton graphs, and it will be shown that there is a bijection between equivalence classes and the postcritically finite Newton maps up to affine conjugacy. This will complete the combinatorial classification of postcritically finite Newton maps.

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