Math 312/ AMS 351 (Fall '22)

HW 7 (due Nov 3)

(Sample Questions for Midterm 2)

1. Let $\pi, \sigma \in \Sigma_5$ be two permutations given by

$$\begin{aligned} \pi &= (12)(345) \\ \sigma &= (13)(24) \end{aligned}$$

   a) Compute $\pi\sigma$ and $\sigma\pi$.

   b) For each of the permutations $\pi, \sigma, \pi\sigma, \sigma\pi$ find the order and sign.

2. §4.3 - 1, 5 (textbook)

3. Let G be a group and let c be a fixed element of G. Define a new operation '$*$' on G by

$$a * b = ac^{-1}b.$$

   Prove that the set G is a group under $*$.

4. Consider the group $U(9)(= \mathbb{Z}_9^*)$ of invertible congruence classes mod 9.

   a) Show that $U(9)$ is cyclic of order 6.

   b) Give an explicit isomorphism $(U(9), \cdot) \cong (\mathbb{Z}_6, +)$.

5. a) Prove that in any finite group, the number of elements of order 3 is even.

   b) Prove that any group of order 12 must contain an element of even order.

   b) Prove that any group of order 12 must contain an element of order 2.

6. Let $G = D(6)$ be the group of symmetries of the regular hexagon.

   0) What is the order of G?

   a) Let R be the set of all rotations in G. Show that R is a subgroup of G. What is the order of R? Is R cyclic?

b) Let $\sigma \in G$ be a reflection. Let $S = \langle \sigma \rangle$. What is the order of S?

c) What are the possible orders $|H|$ of subgroups H in G? Are all the possible orders realized?

d) Is there a cyclic subgroup of order 4 in G?

7. Consider the groups $\mathbb{Z}_2 \times \mathbb{Z}_4$, $D(3)$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_6$, $U(5)$, Σ_3, $\mathbb{Z}_8$, $\mathbb{Z}_4$. Find the odd one out.

8. True or False or Complete

- The positive integers form a group.
- The set of square matrices of size n is a group with respect to matrix ....
- In a group $(ab)^{-1} = \ldots\ldots$
- In an abelian group, $(ab)^2 = a^2b^2$.
- $(\mathbb{Z}_5, \cdot)$ is an abelian group.
- Any group with 6 elements contains an element of order 6.
- A group with 24 elements might contain a subgroup of order 10.
- If G contains an element a of order $|G|$, then G is ....
- The Chinese Remainder Theorem implies that $\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_{24}$.
- The number of invertible elements in $\mathbb{Z}_{24}$ is ....
- A group of oder 4 is always abelian.