Math 313 (Fall '09)

Homework 11

due Dec 10

This is a make-up homework. You should do it only as replacement of a low scoring or missing HW.

1) This exercise tests your understanding of finite fields:
   i) Construct a field with 27 elements and explain why is not possible to have a field with 26 elements.
   ii) The field that you constructed has the form $F(\alpha)$. List the 27 elements and then give the addition and multiplication rules.
   iii) Concretely, compute:
      * $-\alpha =?$
      * $(2\alpha + 1) + (\alpha^2 + 2\alpha + 2) =?$
      * $(2\alpha + 1) \cdot (\alpha^2 + 2\alpha + 2) =?$
      * $\frac{\alpha^2+\alpha+1}{\alpha+1} =?$
   iv) What should be the meaning of $\sqrt{\alpha + 1}$?
   v) We know that $F(\alpha)^*$ is cyclic. Find a generator.

2) This is a continuation of the previous exercise. Let $F(\alpha) = F[x]/\langle x^3 + x^2 + 2x + 1\rangle$ and $F(\beta) = F[x]/\langle 2x^3 + x + 1\rangle$. By general theory, we know that $F(\alpha) \cong F(\beta)$ (explain!). Give an explicit isomorphism $F(\alpha) \cong F(\beta)$.

3) i) Find the minimal polynomial for $\sqrt{-3} + \sqrt{2}$ over $\mathbb{Q}$.
   ii) Make sense of the expression $\sqrt{-3} + \sqrt{2}$ over $\mathbb{Z}_7$ and find its minimal polynomial.

4) i) Let β be a zero of $f(x) = x^5 + 2x + 4$ (it is irreducible). Show that none of $\sqrt{2}$, $\sqrt[3]{2}$, or $\sqrt[4]{2}$ belong to $\mathbb{Q}(\beta)$.
   ii) Prove that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) = \mathbb{Q}(\sqrt[6]{2})$

   Hint: In this exercise, use the degree of the extension.

5) What is the order of the splitting field of $x^5 + x^4 + 1 = (x^2 + x + 1)(x^3 + x + 1)$ over $\mathbb{Z}_2$.

6) Prove that $\pi^2 - 1$ is algebraic over $\mathbb{Q}(\pi^3)$.

7) In which fields does $x^n - x$ have a multiple zero?

8) Prove that the degree of any irreducible factor of $x^8 - x$ over $\mathbb{Z}_2$ is 1 or 3.