Q1.

(a) Compute the following matrix products (if defined).

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(b) Compute the determinants of the matrices A and B below.
For which values of k is A invertible? Explain your answer.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & k \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$.

Q2. Consider the linear system

$$\left|\begin{array}{rrrrr} x+ & & 2z+ & 4w= & -8 \\ & y- & 3z- & w= & 6 \\ 3x+ & 4y- & 6z+ & 8w= & 0 \\ - & y+ & 3z+ & 4w= & -12 \end{array}\right|$$

(a) Write this system in matrix form, $A\vec{x} = \vec{b}$.

(b) Solve the system.

(c) What is the rank of the matrix A? (A is the coefficient matrix for the system you just solved.)
Explain your answer.

(d) What can you say about the number of solutions of $A\vec{x} = \vec{0}$?
Explain your answer.

(e) What is $\det A$? Explain how you found it.

Q3. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(a) Do these vectors span $\mathbb{R}^5$? Explain your answer.

(b) Are these vectors linearly independent? Explain your answer.

(c) What is the dimension of the subspace L spanned by these vectors? Find a basis of L. Explain how you found your answers.

(d) Consider the vectors $\vec{w}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Decide which of them (if any) belong to the subspace L. Explain your answers.

(e) The vector $\vec{x} = \begin{bmatrix} 5 \\ -5 \\ -5 \\ -5 \\ -5 \end{bmatrix}$ belongs to L. Find its coordinates in the basis of L you gave in part (b).

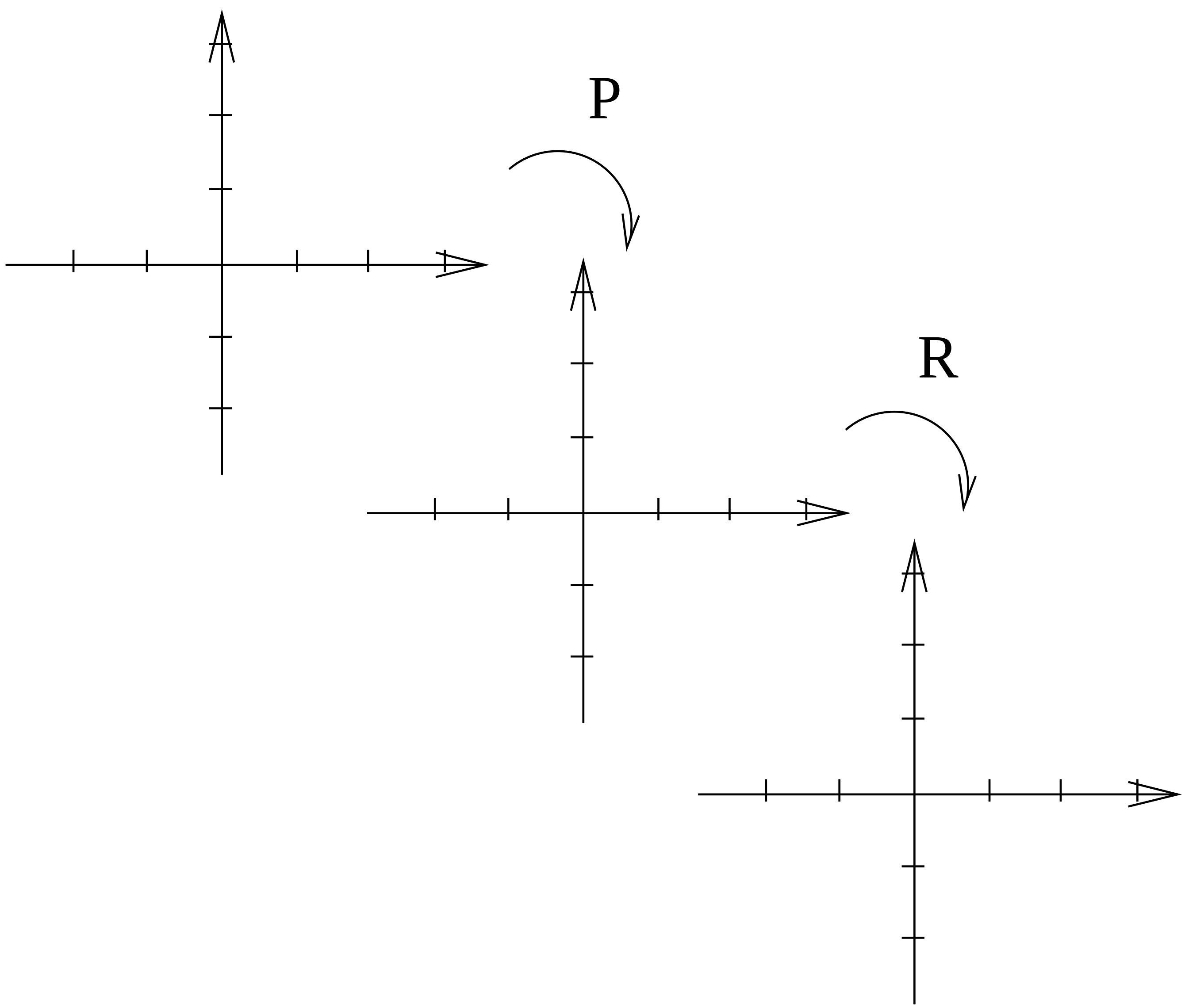
Q4. This a question about a linear transformation of the plane. Using the geometry of the transformation will be very helpful and will give the quickest solutions. You are welcome to use any other methods if you prefer.

(a) Consider a linear transformation T which is given by the orthogonal projection P onto the y-axis followed by rotation R by $180°$ counterclockwise about the origin. Use the axes below to show how the transformation T works. (Please label everything and make a clear picture.)

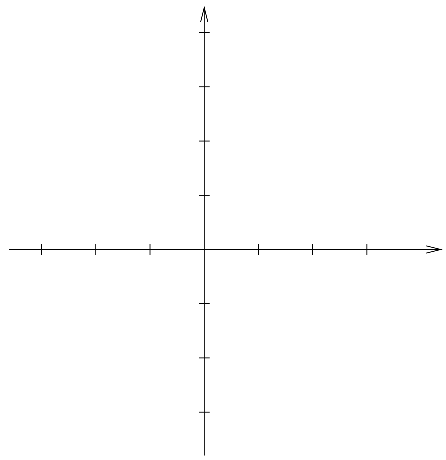
In the top left picture, draw vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

the middle picture, draw the vectors $w_1 = Pv_1$, $w_2 = Pv_2$, $w_3 = Pv_3$.

In the bottom right picture, draw the vectors $u_1 = Rw_1$, $u_2 = Rw_2$, $u_3 = Rw_3$.
(Notice that $T = R \circ P$, so $u_1 = RPw_1 = Tv_1$, and similarly for the other two vectors.)



(b) Find kernel and image of the tranformation T. Explain your answer. Draw $Ker\,T$ and $Im\,T$ clearly in the plane below.



(c) Write the matrix A of the transformation T considered above.
Explain how you got your answer.

(d) Is the matrix A invertible? What is the rank of A? Is the transformation T an isomorphism? Explain your answers.

Q5. (a) Let L be the subspace spanned by the vectors $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. Apply the orthogonalization process to these vectors to find an orthonormal basis for L.

(BONUS QUESTION: can you see another orthonormal basis for L? You can find a simple one without using any process. Justify your answer.)

(b) Find the orthogonal projection of the vector $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ onto L. (You can use any method.)

(c) For the vectors $\vec{u}_1, \vec{u}_2$ that you found in (a), find a vector $\vec{w}$ such that $(\vec{w}, \vec{u}_1, \vec{u}_2)$ forms an orthonormal basis for $\mathbb{R}^3$. Explain how you found your answer. (You can use any method.)

Q6. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Find eigenvalues of A.

(b) For each eigenvalue that you found above, find a corresponding eigenvector. What is the dimension of each eigenspace?

(c) Can you diagonalize the matrix A? If so, what is the corresponding diagonal matrix? Explain your answers.