Math 122 (Fall '12)

Review Questions for Final (Partial Solutions)

Disclaimer: You should use these solutions only **after** you tried to solve the exercises yourself. Be aware that there might be typos and small computational mistakes. Please make sure that you understand the arguments, and don't focus exclusively on the answer.

Part I - Fundamental Questions

1. Solve the following equations:

i. Linear Equations: $2(x+3) = 5$

Answer: $x = -\frac{1}{2}$

ii. Quadratic Equations: $2x^2 + 5x = 3$

Answer: Factor as $(2x-1)(x+3) = 0$. Thus the *two solutions* are $x = \frac{1}{2}$ and $x = -3$.

Alternatively, you could apply the general formula for quadratic equations of type

$$ax^2 + bx + c = 0.$$

Namely, the solutions are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ for $\Delta = b^2 - 4ac$. Here $\Delta = 49$. Thus $x_{1,2} = \frac{-5 \pm 7}{4}$, giving the solutions -3 and $\frac{1}{2}$ as before.

iii. Equations involving exponential functions: $2 \cdot e^x = 3 \cdot 2^x$

Answer: Take ln of both sides. Get

$$\ln 2 + x = \ln 3 + x \ln 2,$$

Thus,

$$\begin{aligned} x(1-\ln 2) &= \ln 3 - \ln 2 \\ x &= \frac{\ln 3 - \ln 2}{1 - \ln 2} \end{aligned}$$

iv. Equations involving logarithms: $2\ln x = \ln(3x) + 5$

Answer: Get

$$2\ln x = \ln x + \ln 3 + 5$$

Thus

$$\ln x = \ln 3 + 5 = \ln 3 + \ln e^5 = \ln 3e^5$$

Exponentiate both sides and get the final answer

$$x = 3e^5.$$

v. Equations involving power functions: $x^{100} = 2^{99} \cdot \sqrt{x^{101}}$

Answer: Write the equation in power form

$$x^{100} = 2^9 9 \cdot x^{\frac{101}{2}}$$

Thus

$$\begin{aligned} x^{100-\frac{101}{2}} &= 2^{99} \\ x^{\frac{99}{2}} &= 2^{99} \end{aligned}$$

Raising both sides to power $\frac{2}{99}$ gives the final answer

$$x = 2^2 = 4$$

2.

i. $3x^2 + 2e^x + \sqrt{x^3} + \frac{4}{x} - \ln x$ (basic rules)

Answer:

$$6x + 2e^x + \frac{3}{2}\sqrt{x} - \frac{4}{x^2} - \frac{1}{x}$$

ii. $(x^2 + 1)\ln x$ (product rule)

Answer:

$$2x\ln x + \frac{x^2+1}{x}$$

iii. $\ln(x^2 + 1)$ (chain rule)

Answer:

$$\frac{2x}{x^2+1}$$

iv. $\frac{x}{x^2+1}$ (quotient rule)

Answer:

$$\frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

v. $e^{x\ln x} + xe^{2\ln x+3}$ (both chain and product rules)

Answer:

$$e^{x\ln x}(\ln x + 1) + e^{2\ln x+3} + xe^{2\ln x+3} \cdot \frac{2}{x}$$

3. Find an antiderivative for each of the following functions

i. $3x^2 + 2e^x + \sqrt{x^3} + \frac{4}{x}$ (basic rules)

Answer:

$$x^3 + 2e^x + \frac{2}{5}x^{\frac{5}{2}} + 4\ln x + C$$

ii. $\frac{\sqrt{\ln x}}{x}$ (substitution) **Answer:**

$$\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}$$

Thus

$$\int \frac{\sqrt{\ln x}}{x}\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$$

iii. $x^2 e^{x^3+1}$ (substitution)

Answer:

$$\begin{aligned} u &= x^3 + 1 \\ du &= 3x^2 dx \\ \frac{du}{2} &= x^2 dx \end{aligned}$$

Thus,

$$\int x^2 e^{x^3+1}\, dx = \int e^u \frac{du}{3} = \frac{e^u}{3} = \frac{e^{x^3+1}}{3} + C$$

iv. $x(2x^2 + 3)^{100}$ (substitution)

Answer:

$$\begin{aligned} u &= 2x^2 + 3 \\ du &= 4x dx \end{aligned}$$

Get

$$\int u^100 \frac{du}{4} = \frac{u^{101}}{4 \cdot 101} = \frac{(2x^2+3)^{101}}{404} + C$$

v. $x^2(x^5 - x^6)$ (expand, basic rules)

Answer:

$$\int x^2(x^5 - x^6)\ dx = \int x^7 - x^8\ dx = \frac{x^8}{8} - \frac{x^9}{9} + C$$

4. Questions related to the fundamental theorem of calculus

A. Compute the definite integrals

i) $\int_0^1 2x - e^x\ dx$

Answer:

$$\int_0^1 2x - e^x\ dx = (x^2 - e^x)\ |_0^1 = (1 - e^1) - (0 - e^0) = 2 - e$$

ii) $\int_1^4 x\sqrt{x^2+1}\ dx$

Answer: We compute first the antiderivative of $x\sqrt{x^2+1}$ by using the substitution

$$\begin{aligned} u &= x^2 + 1 \\ du &= 2xdx \end{aligned}$$

Thus

$$\int x\sqrt{x^2+1}\ dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}(x^2+1)^{\frac{3}{2}}$$

Now, go back to the original question and get

$$\int_1^4 x\sqrt{x^2+1}\ dx = \frac{1}{3}(x^2+1)^{\frac{3}{2}}\ |_1^4 = \frac{1}{3}\left(17^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

B. Is it true that

i) $\int xe^x \, dx = (x-1)e^x + C$?

Answer: Compute the derivatives of both sides. By the Fundamental Theorem of Calculus, on the LHS we get the original function, i.e. xe^x, on the RHS we need to compute the derivative of $(x-1)e^x$. In other words, the original question is equivalent to the equality

$$xe^x = (x-1)e^x)'???$$

Now

$$(x-1)e^x)' = e^x + (x-1)e^x = xe^x$$

So indeed the above equality holds, so the final answer is TRUE.

ii) $\int x \ln x \, dx = (x-1)\ln x + C$? **Answer:**

$$x \ln x = ((x-1)\ln x)'???$$

But now,

$$((x-1)\ln x)' = \ln x + \frac{x-1}{x}$$

Thus, the final answer is FALSE.

Part II - Applications of derivatives/integrals

7. (Integrals) Estimation of the integral using Riemann Sums (see [SFinal](8)]. For instance:

i) Estimate $\int_0^2 x^2 dx$ using 4 division points and using

Answer: We divide the interval in 4 pieces. Thus the division points are 0, 0.5, 1, 1.5 and 2. The size of the small intervals is $\Delta x = 0.5$.

- the left end-point

  Answer:

$$f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x =$$

$$= 0^2 \cdot 0.5 + 0.5^2 \cdot 0.5 + 1^2 \cdot 0.5 + 1.5^2 \cdot 0.5 = 1.75$$

- the right end-point

  Answer:

$$f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x = 3.75$$

- the mid-point

  Answer:

$$f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x = 2.625$$

Which is one is the most accurate method? Which one is clearly an underestimate? Which one is clearly an overestimate? Sketch a graph

Answer: Underestimate: Left. Overestimate: Right. Best Estimate: Midpoint.

ii) What is a reasonable values for $\int_1^3 f(x)\,dx$ if you are given

x	0	0.5	1	1.5	2	2.5	3
f(x)	1	1.2	1.5	2	1.4	1.1	0.7

? **Answer:** Here you could use either left or right Riemann sums to estimate the integral (either one of them is acceptable for the final). We get

$$\int_1^3 f(x)\,dx \sim f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x =$$

$$= 1.5 \cdot 0.5 + 2 \cdot 0.5 + 1.4 \cdot 0.5 + 1.1 \cdot 0.5 = 3$$

8. (Integrals) Compute areas (see [SFinal](7)). For instance

i) Find the area below the graph for $\sqrt{x}$ for $x \in [1, 9]$. **Answer:**

$$\text{Area} = \int_1^9 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\,|_1^9 = \frac{2}{3}(9^{\frac{3}{2}} - 1) = \frac{52}{3}$$

Note $x^{\frac{3}{2}} = x\sqrt{x}$. Thus $9^{\frac{3}{2}} = 9\sqrt{9} = 27$.

ii) Find the area between the graphs of the functions $f(x) = 3x - 2$ and $g(x) = x^2$. (**Note:** the original equations do not work; there is a single point of intersection.)

Answer: Note that you are not given the interval over which to integrate. Thus, you need to find the intersection points for the 2 curves (draw a graph). In other words, you need to solve

$$3x - 2 = x^2$$

One gets, $x = 1$ and $x = 2$. Thus, we need to compute

$$\text{Area} = \int_1^2 (3x-2)-x^2\,dx = \frac{3}{2}x^2 - 2x - \frac{1}{3}x^3\,|_1^2 = (\frac{12}{2} - 4 - \frac{8}{3}) - (\frac{3}{2} - 2 - \frac{1}{3})$$

Thus,

$$\text{Area} = 4 - \frac{3}{2} - \frac{7}{3} = \frac{1}{6}$$

(**Again, I warn you that the numerics might be wrong. Please check the answers carefully.**)