Math 534 (Fall '17)

Midterm 1

September 26, 2017

Note: Choose 5 out 6 exercises.

1. Classify all groups of order 50 and 51 (the easier case) respectively.

2. i) Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

ii) Prove the existence of a p-Sylow subgroup.

3. We know that any inner automorphism of G is realized by acting with G on G by conjugation. Prove that we can embed G as a normal subgroup in some group G' such that the morphism $G' \to \mathrm{Aut}(G)$ given by conjugation in G' is surjective.
   Hint: Is there a natural semi-direct product that you might consider in this situation? Think about what you need to define a semi-direct product.

4. Prove that any p-group is solvable.

5. What is the smallest nonabelian simple group?
   i) Make a guess G (you don't need to prove that your guess is simple);
   ii) Argue that any group of smaller order is either abelian or non-simple;
   iii) Argue that there is no other simple group of the same order as your guess G.

6. Let H be a subgroup of a finite abelian group G. Show that G has a subgroup isomorphic to G/H. Give an example to show that this does not imply that G is a semi-direct product of H and G/H.