Calculus IV with Applications MAT303
Solutions to Practice Problems for the Final

5.1, 18. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, P(t) = \begin{bmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{bmatrix}, \mathbf{f}(t) = \mathbf{0}$.

5.1, 26. $W = \begin{bmatrix} 2e^t & -2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & e^{3t} & e^{5t} \end{bmatrix} = 16e^te^{3t}e^{5t} = 16e^{9t} \neq 0$. General solution: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3}$.

5.2, 6. $A = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 12 = 0$, so $\lambda = 3, 4$. Eigenvector for $\lambda = 3$, $\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$. For $\lambda = 4$, similarly $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. General solution: $c_1 \begin{bmatrix} 5 \\ -6 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$. $x_1 = 5c_1e^{3t} + c_2e^{4t}, x_2 = -6c_1e^{3t} - c_2e^{4t}$. For given initial values, $5c_1 + c_2 = 1, -6c_1 - c_2 = 0$, i.e. $c_1 = -1, c_2 = 6$.

5.2, 9. $A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + 16 = 0$, so $\lambda = \pm 4i$. Eigenvector for $\lambda = 4i$, $\begin{bmatrix} 2-4i & -5 \\ -6 & -2-4i \end{bmatrix} \mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$. Particular complex solution: $\begin{bmatrix} 5 \\ 2-4i \end{bmatrix} e^{4it} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix} (\cos 4t + i \sin 4t) = \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + i \begin{bmatrix} 5\sin 4t \\ -4\cos 4t + 2\sin 4t \end{bmatrix}$. General solution: $c_1 \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin 4t \\ -4\cos 4t + 2\sin 4t \end{bmatrix}$. $x_1 = 5c_1 \cos 4t + 5c_2 \sin 4t, x_2 = c_1(2\cos 4t + 4\sin 4t) + c_2(-4\cos 4t + 2\sin 4t)$. For given initial values,

5.2, 23. $A = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{vmatrix} = (3 - \lambda)((3-\lambda_(-3-\lambda) + 5) = (3-\lambda)(\lambda^2 - 4) = 0$. $\lambda = -2, 2, 3$. Eigenvectors: for $\lambda = -2$, $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$; for $\lambda = 2$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; for $\lambda = 3$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. General solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$.

5.4, 6. Eigenvalues: $\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = 0$, so $\lambda = 5$ (multiplicity 2). Eigenvector(s) for $\lambda = 5$: $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \mathbf{v} = 0$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Defect of $\lambda = 1$. Generalized eigenvectors: $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so can choose $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\mathbf{v}$ is an eigenvector, gives particular solution $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t}$. $\mathbf{w}$ is not an eigenvector, so use $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \mathbf{w} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$ to obtain particular solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} te^{5t}$. Answer $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} te^{5t} \right)$.

5.4, 8. Eigenvalues: $\begin{vmatrix} 25-\lambda & 12 & 0 \\ -18 & -5-\lambda & 0 \\ 6 & 6 & 13-\lambda \end{vmatrix} = (13-\lambda)(\lambda^2 - 20\lambda + 91) = 0$, $\lambda = 7, 13, 13$. Eigenvector for $\lambda = 7$, $\begin{bmatrix} 18 & 12 & 0 \\ -18 & -12 & 0 \\ 6 & 6 & 6 \end{bmatrix} \mathbf{v} = \mathbf{0}$, so $v_1 + v_2 + v_3 = 0$ (3rd equation), $3v_1 + 2v_2 = 0$ (1st or 2nd equations), get $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$ and a particular solution $\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} e^{7t}$. Eigenvector(s) for $\lambda = 13$, $\begin{bmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{bmatrix} \mathbf{v} = \mathbf{0}$, get $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; particular solutions: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{13t}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{13t}$. Answer: $\mathbf{x} = c_1 \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{13t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{13t}$.

5.4, 17. Eigenvalues: $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 18 & 7-\lambda & 4 \\ -27 & -9 & -5-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 2\lambda + 1) = (\lambda - 1)^3$, so $\lambda = 1$ with multiplicity 3. Eigenvector(s): $\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix} \mathbf{v} = \mathbf{0}$, so $9v_1 + 3v_2 + 2v_3 = 0$ (2nd or 3rd eq-n). Eigenvectors: $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix}$ giving particular solutions $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^t$ and $\begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix} e^t$. Defect of $\lambda = 1$. Generalized eigenvectors: $\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix}^2 \mathbf{w} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{w} = \mathbf{0}$. So, every vector is a generalized eigenvector. We need three linearly independent ones, so choose the two original eigenvectors and one vector linearly independent from them, e.g. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Use $\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 0 \\ 18 \\ -27 \end{bmatrix}$ to obtain the particular solution $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 18 \\ -27 \end{bmatrix} te^t$.

Answer: $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 18 \\ -27 \end{bmatrix} te^t \right)$.

5.5, 5. First, find the general solution: $A = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}$. Eigenvalues: $\lambda = \pm 3i$. Eigenvector for $\lambda = 3i$: $\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix} \mathbf{v} = \mathbf{0}$. $\mathbf{v} = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix}$. Particular complex solution: $\begin{bmatrix} -2 \\ 3+3i \end{bmatrix} e^{3it} = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix} (\cos 3t + i \sin 3t) = \begin{bmatrix} -2\cos 3t \\ 3cos3t - 3\sin 3t \end{bmatrix} + i \begin{bmatrix} -2\sin 3t \\ 3\sin 3t + 3\cos 3t \end{bmatrix}$. General solution: $\mathbf{x} = c_1 \begin{bmatrix} -2\cos 3t \\ 3cos3t - 3\sin 3t \end{bmatrix} + c_2 \begin{bmatrix} -2\sin 3t \\ 3\sin 3t + 3\cos 3t \end{bmatrix}$. Fundamental matrix: $\Phi(t) = \begin{bmatrix} -2\cos 3t & -2\sin 3t \\ 3\cos 3t - 3\sin 3t & 3\sin 3t + 3\cos 3t \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} -2 & 0 \\ 3 & 3 \end{bmatrix}$, $\Phi(0)^{-1} = -\frac{1}{6} \begin{bmatrix} 3 & 0 \\ -3 & -2 \end{bmatrix}$. Answer: $\begin{bmatrix} -2\cos 3t & -2\sin 3t \\ 3\cos 3t - 3\sin 3t & 3\sin 3t + 3\cos 3t \end{bmatrix} \cdot -\frac{1}{6} \begin{bmatrix} 3 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2\cos 3t & -2\sin 3t \\ 3\cos 3t - 3\sin 3t & 3\sin 3t + 3\cos 3t \end{bmatrix} \cdot \frac{-1}{6} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3\cos 3t - \sin 3t \\ -3\cos 3t + 6\sin 3t \end{bmatrix}$.

5.5, 25. $\mathbf{X} = \exp\left(\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} t\right) = \exp\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t\right) \exp\left(\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t\right) = e^{2t} \exp\left(\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t\right) = e^{2t}\left(\mathbf{I} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t\right)$ (higher powers of $\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$ are zero) $= e^{2t} \begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}$.
$\mathbf{x} = \mathbf{X} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 4e^{2t} + 35te^{2t} \\ 7e^{2t} \end{bmatrix}$.

5.5, 26. $\mathbf{X} = \exp\left(\begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix} t\right) = \exp\left(\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} t\right) \exp\left(\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t\right) = e^{7t} \exp\left(\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t\right) = e^{7t}\left(\mathbf{I} + \begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t\right)$ (higher powers of $\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix}$ are zero) $= e^{7t} \begin{bmatrix} 1 & 0 \\ 11t & 1 \end{bmatrix} = \begin{bmatrix} e^{7t} & 0 \\ 11te^{7t} & e^{7t} \end{bmatrix}$.
$\mathbf{x} = \mathbf{X} \begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 5e^{7t} \\ e^{7t}(55t - 10) \end{bmatrix}$.

6.1, 5. $1 - y^2 = 0, x + 2y = 0$. Hence, $y = \pm 1$ and, for $y = 1$, $x = -2$; for $y = -1$, $x = 2$. Critical points: $(-2, 1), (2, -1)$.

6.1, 15. $x' = -2x$, thus $x = ae^{-2t}$; $y' = -y$, thus $y = be^{-t}$. As $t \to \infty$, $x \to 0, y \to 0$. Asymptotically stable.

6.1, 17. $x' = y, y' = -x$, thus $x'' = y' = -x$. $x'' + x = 0$. Characteristic equation: $r^2 + 1 = 0$. $x = c_1 \cos t + c_2 \sin t$ and $y = c_1 \sin t - c_2 \cos t$. Stable.

6.2, 5. $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$. $\lambda_1 = \lambda_2 = -1 < 0$. Asymptotically stable.

6.2, 16. Linearization: $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = 0$. $\lambda = 2 \pm i$. Real parts of λ are positive. Unstable.

6.2, 23. Linearization: $\begin{bmatrix} 2+3x^2 & -5 \\ 4 & -6+4y^3 \end{bmatrix}$. At $(0,0)$, $\begin{bmatrix} 2 & -5 \\ 4 & -6 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 2-\lambda & -5 \\ 4 & -6-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 8 = 0$. $\lambda = -2 \pm 2i$. Real parts are negative. Asymptotically stable.

6.2, 26. Linearization: $\begin{bmatrix} 3-2x & -2-2y \\ 2-2x & -1+4y^3 \end{bmatrix}$. At $(0,0)$, $\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 3-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0$. $\lambda = 1 > 0$. Unstable.