INDUCTION AND PIGEONHOLE

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Problem 1. Determine the maximum value of $\sin A_1 ... \sin A_n$ given $\tan A_1 ... \tan A_n = 1$.

Problem 2. Given positive numbers $x_1, ..., x_n$ whose product is 1, prove $\sum x_i \leqslant \sum x_i^2$.

Problem 3. Suppose $x_1, ..., x_n$ are positive real numbers. Show that

$$\frac{x_1^2}{x_1 + x_2} + \frac{x_2^2}{x_2 + x_3} + \cdots + \frac{x_n^2}{x_n + x_1} \leqslant \frac{x_1 + \cdots + x_n}{2}.$$

Problem 4. Prove the 'logarithmic mean' inequality, for $a > b > 0$,

$$\sqrt{ab} \leqslant \frac{a - b}{\ln a - \ln b} \leqslant \frac{a + b}{2}.$$

Problem 5. Let A be a positive real number. What are the possible values of $\sum_{n=1}^{\infty} x_n^2$ given the positive real numbers $x_1, x_2, ...$ satisfy $\sum_{n=1}^{\infty} x_n = A$?

Problem 6. Determine the greatest possible value for the sum $\sum_{i=1}^{10} \cos 3x_i$ for the real numbers $x_1, ..., x_{10}$ satisfying $\sum_{i=1}^{10} \cos x_i = 0$.

Problem 7. For what pairs of positive real numbers (a, b) does the improper integral shown converge?

$$\int_b^{\infty} \left(\sqrt{\sqrt{x + a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x - b}} \right) dx$$

Problem 8. Let (a_n) and (b_n) be two sequences of positive numbers. Prove that the following two statements are equivalent

(1) There is a sequence of positive numbers c_n such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ converge.
(2) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

Problem 9. Let a_n be a sequence of positive numbers such that $\sum a_n$ converges. Prove that $\sum_{n=1}^{\infty} (a_1 ... a_n)^{\frac{1}{n}}$ converges.

Problem 10. Let $a_1, ..., a_n$ be real numbers greater than -1, all of the same sign. Prove that

$$(1 + a_1) ... (1 + a_n) \geqslant 1 + a_1 + ... + a_n.$$

Problem 11. Let $a_1, ..., a_{n+1}$ be positive real numbers with $a_1 = a_{n+1}$. Prove

$$\sum_{i=1}^{n} \left(\frac{a_{i+1}}{a_i} \right)^n \geqslant \sum_{i=1}^{n} \frac{a_i}{a_{i+1}}.$$