CONGRUENCES

ROBERT HOUGH

Problem 1. How many coefficients of the polynomial

$$P_n(x_1, ..., x_n) = \prod_{1 \leqslant i < j \leqslant n} (x_i + x_j)$$

are odd?

Problem 2. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

Problem 3. Suppose that $f(x)$ and $g(x)$ are polynomials with $f(x)$ not identically 0 taking integers to integers such that for all $n \in \mathbb{Z}$, either $f(n) = 0$ or $f(n)|g(n)$. Show that $f(x)|g(x)$, i.e. there is a polynomial with rational coefficients $h(x)$ such that $g(x) = f(x)h(x)$.

Problem 4. Do there exist 1000000 consecutive integers each of which contains a repeated prime factor?

Problem 5. A positive integer n is powerful if for every prime p dividing n, p^2 divides n. Show that for any $k \geqslant 1$ there exist k consecutive integers, none of which is powerful.

Problem 6. Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

Problem 7. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

Problem 8. What is the units digit of

$$\left[\frac{10^{20000}}{10^{100} + 3} \right]?$$

Problem 9. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, ..., [n/i]\}.$$

Problem 10. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geqslant 0$. Show that u_n is an integer for all n.