SEQUENCES AND SERIES

ROBERT HOUGH

Problem 1. Define the sequence $(a_n)_{n\geqslant 0}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \qquad n \geqslant 0.$$

Prove that n divides a_n for all $n \geqslant 1$.

Problem 2. The sequence $a_0, a_1, a_2, ...$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all non-negative integers m and n with $m \geqslant n$. If $a_1 = 1$, determine a_n.

Problem 3. Find the general term of the sequence given by $x_0 = 3, x_1 = 4$, and

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2}, \qquad n \geqslant 2.$$

Problem 4. Compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1}.$$

Problem 5. Prove that the sequence $(a_n)_{n\geqslant 1}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1), \qquad n \geqslant 1$$

has a limit.

Problem 6. Let t and ϵ be real numbers with $|\epsilon| < 1$. Prove that the equation $x - \epsilon \sin x = t$ has a unique real solution.

Problem 7. Let p be a real number, $p \neq 1$. Compute

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}.$$

Problem 8. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function with the property that for any $x > 0$, $\lim_{n\to\infty} f(nx) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

Problem 9. Does the series $\sum_{n=1}^{\infty} \sin \pi\sqrt{n^2+1}$ converge?

Problem 10. The number q ranges over all possible powers with both the base and exponent positive integers greater than 1, assuming each value only once. Prove that

$$\sum_{q} \frac{1}{q-1} = 1.$$

Problem 11. Let

$$a_n = \frac{4n + \sqrt{4n^2-1}}{\sqrt{2n+1} + \sqrt{2n-1}}, \qquad n \geqslant 1.$$

Prove that $a_1 + a_2 + \cdots + a_{40}$ is a positive integer.

Problem 12. Evaluate in closed form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m!n!}{(m+n+2)!}.$$