POLYNOMIALS

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Problem 1. Let $P(x)$ be a polynomial of degree n. Given $P(k) = \frac{k}{k+1}$, $k = 0, 1, 2, ..., n$ find $P(m)$ for $m > n$.

Problem 2. Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even function if and only if there exists a polynomial $Q(x)$ with complex coefficients satisfying $P(x) = Q(x)Q(-x)$.

Problem 3. Prove that for every positive integer n,

$$\tan\frac{\pi}{2n+1}\tan\frac{2\pi}{2n+1}\cdots\tan\frac{n\pi}{2n+1} = \sqrt{2n+1}.$$

Problem 4. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geqslant 3$. Given $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all of the roots are real, find the remaining coefficients.

Problem 5. Let $P_n(x) = (x^n - 1)(x^{n-1} - 1)\cdots(x-1)$, $n \geqslant 1$. Prove that for $n \geqslant 2$, $P_n'(x)$ is divisible by $P_{\lfloor n/2 \rfloor}(x)$ in the ring of polynomials with integer coefficients.

Problem 6. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros $x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n$ are real. Prove that

$$P'\left(\frac{x_1 + x_2}{2}\right) \cdot P'\left(\frac{x_{n-1} + x_n}{2}\right) \neq 0.$$

Problem 7. Let $P(z)$ be a polynomial of degree $2n$, all of whose zeros have absolute value 1 in the complex plane. Let $g(z) = \frac{P(z)}{z^n}$. Show that all of the roots of $g'(z)$ have absolute value 1.

Problem 8. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients, with $a_0 \neq 0$, and with the property that there exists an $m \geqslant 1$ with $\left|\frac{a_m}{a_0}\right| \geqslant \binom{n}{m}$. Prove that $P(x)$ has a zero of absolute value less than 1.

Problem 9. Prove that for every positive integer n, the polynomial $P(x) = x^{2^n} + 1$ is irreducible over $\mathbb{Z}[x]$.

Problem 10. Prove that for any distinct integers $a_1, a_2, ..., a_n$ the polynomial

$$P(x) = (x - a_1)(x - a_2)\cdots(x - a_n) - 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

Problem 11. Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + 3x^{p-3} + \cdots + (p-1)x + p$$

is irreducible in $\mathbb{Z}[x]$.