NUMBER THEORY

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Problem 1. Let k be a positive integer. The sequence $(a_n)_n$ is defined by $a_1 = 1$ and for $n \geqslant 2$, a_n is the nth positive integer greater than a_{n-1} that is congruent to n modulo k. Find a_n in closed form.

Problem 2. Prove that for any positive integer n,

$$\lfloor \sqrt{n} \rfloor = \left\lfloor \sqrt{n} + \frac{1}{\sqrt{n} + \sqrt{n+2}} \right\rfloor.$$

Problem 3. For p and q co-prime positive integers, prove the reciprocity law

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \cdots \left\lfloor \frac{(p-1)q}{p} \right\rfloor.$$

Problem 4. Let p be a prime number. Prove that there are infinitely many multiples of p whose last ten digits are all distinct.

Problem 5. The last four digits of a perfect square are all equal. Prove that they are all 0.

Problem 6. Given a positive integer $n > 1000$, add the residues of 2^n modulo each of the numbers $1, 2, 3, ..., n$. Prove that this sum is greater than $2n$.

Problem 7. Let $(x_n)_n$ be a sequence of positive integers satisfying the recurrence relation $x_{n+1} = 5x_n - 6x_{n-1}$. Prove that infinitely many of the terms are composite.

Problem 8. Prove that for any positive integer n other than 2 or 6, $\phi(n) \geqslant \sqrt{n}$.

Problem 9. Prove that there are infinitely many even positive integers m for which the equation $\phi(n) = m$ has no solution.

Problem 10. Prove that for every n, there exist n consecutive integers each of which is divisible by two different primes.

Problem 11. Let a and b be positive integers. For a non-negative integer n let $s(n)$ be the number of non-negative integer solutions to the equationa $ax + by = n$. Prove that the generating function of the sequence $(s(n))_n$ is $f(x) = \frac{1}{(1-x^a)(1-x^b)}$.

Problem 12. Find a solution to the Diophantine equation

$$x^2 - (m^2+1)y^2 = 1$$

where m is a positive integer.