MULTIVARIABLE CALCULUS

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Problem 1. Prove that for $|x| < 1$,

$$\arcsin x = \sum_{k=0}^{\infty} \frac{1}{2^{2k}(2k+1)} \binom{2k}{k} x^{2k+1}.$$

Problem 2. Given n points in the plane, suppose there is a unique line that minimizes the sum of the distances from the points to the line. Prove that the line passes through two of the points.

Problem 3. Of all of the triangles circumscribed about a given circle, find the one with the smallest area.

Problem 4. Find the integral of the function $f(x, y, z) = \frac{x^4+2y^4}{x^4+4y^4+z^4}$ over the unit ball $\{(x, y, z) : x^2 + y^2 + z^2 \leqslant 1\}$.

Problem 5. Compute the integral $\iint_D x dx dy$ over

$$D = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0, 1 \leqslant xy \leqslant 2, 1 \leqslant \frac{y}{x} \leqslant 2\}.$$

Problem 6. Let $a_1 \leqslant a_2 \leqslant ... \leqslant a_n = m$ be positive integers. Let b_k be the number of a_i for which $a_i \geqslant k$. Prove that

$$a_1 + \cdots + a_n = b_1 + \cdots + b_m.$$

Problem 7. Show that for $a, b > 0$,

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}.$$

Problem 8. Let $|x| < 1$. Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{1}{t} \ln(1-t) dt.$$

Problem 9. Compute the flux of the vector field

$$F(x, y, z) = x(e^{xy} - e^{zx})i + y(e^{yz} - e^{xy})j + z(e^{zx} - e^{yz})k$$

across the upper hemisphere of the unit sphere.

Problem 10. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (F_1(x, y), F_2(x, y))$ be a vector field, and let $G : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function whose first two variables are x and y, and third is t, the time. Assume that for any rectangular surface D bounded by the curve C,

$$\frac{d}{dt} \iint_D G(x, y, t) dx dy = -\oint F \cdot dR.$$

Prove that

$$\frac{\partial G}{dt} + \frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y} = 0.$$

Problem 11. Let f and g be real valued functions defined for all real numbers and satisfying

$$f(x+y)+f(x-y)=2f(x)g(y)$$

for all x, y. Prove that if f is not identically zero, and if $|f(x)| \leqslant 1$ for all x then $|g(y)| \leqslant 1$ for all y.

Problem 12. Find all continuous $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(xy) = xf(y) + yf(x), \qquad x, y \in \mathbb{R}.$$