INEQUALITIES

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Problem 1. If a, b, c are positive numbers, prove

$$9a^2b^2c^2 \leqslant (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2).$$

Problem 2. Let $P(x)$ be a polynomial with real coefficients. Prove that $\sqrt{P(a)P(b)} \geqslant P(\sqrt{ab})$ for all positive a, b.

Problem 3. Let $P(z)$ be a polynomial with real coefficients whose roots can be covered by a disk of radius R. Prove that for any real number k, the roots of $nP(z) - kP'(z)$ can by covered by a disk of radius $R+|k|$, where n is the degree of P and P' is the derivative.

Problem 4. Prove that the positive real numbers a, b, c are the side lengths of a triangle if and only if

$$a^2 + b^2 + c^2 < 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

Problem 5. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be non-negative numbers. Show that

$$(a_1a_2\cdots a_n)^{\frac{1}{n}} + (b_1b_2\cdots b_n)^{\frac{1}{n}} \leqslant ((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n))^{\frac{1}{n}}.$$

Problem 6. Show that all real roots of $x^5 - 10x + 35$ are negative.

Problem 7. Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1 + \cdots + a_n < 1$. Prove that

$$\frac{a_1a_2\cdots a_n(1-(a_1+\cdots+a_n))}{(a_1+\cdots+a_n)(1-a_1)\cdots(1-a_n)} \leqslant \frac{1}{n^{n+1}}.$$

Problem 8. Given a positive integer n, find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

subject to the constraint the x_i are distinct positive integers.

Problem 9. Assume that all of the zeros of the polynomial $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ are real and positive. Show that if there exist $1 \leqslant m < p \leqslant n$ such that $a_m = (-1)^m\binom{n}{m}$ and $a_p = (-1)^p\binom{n}{p}$, then $P(x) = (x-1)^n$.

Problem 10. Let $a_1, ..., a_n, b_1, ..., b_n$ be real numbers such that

$$(a_1^2 + \cdots + a_n^2 - 1)(b_1^2 + \cdots + b_n^2 - 1) > (a_1b_1 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + \cdots + a_n^2 > 1$ and $b_1^2 + \cdots + b_n^2 > 1$.