INDUCTION AND PIGEONHOLE

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Problem 1. Prove $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is irrational.

Problem 2. Prove no nine consecutive positive integers can be partitioned into two sets of equal product.

Problem 3. Show that there does not exist a function $f : \mathbb{Z} \to \{1, 2, 3\}$ satisfying $f(x) \neq f(y)$ when $|x - y| \in \{2, 3, 5\}$.

Problem 4. Prove that the Fibonacci sequence $F_0 = 0, F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ satisfies for all n, $F_{2n+1} = F_{n+1}^2 + F_n^2$.

Problem 5. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies for all x_1, x_2, $f\left(\frac{x_1+x_2}{2}\right) = \frac{f(x_1)+f(x_2)}{2}$. Prove that for all $n \geqslant 1$ and $x_1, ..., x_n$,

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) = \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

Problem 6. Prove that for non-negative $a_1, ..., a_n$,

$$(1 + a_1)\cdots(1 + a_n) \geqslant \left(1 + (a_1 \cdots a_n)^{\frac{1}{n}}\right)^n.$$

Problem 7. Prove that for any integer $m \geqslant 1$ there is a Fibonacci number divisible by m.

Problem 8. Let $n(r)$ be the number of integer lattice points on a circle of radius $r > 1$. Prove that $n(r) < 2\pi r^{\frac{2}{3}}$.

Problem 9. Four congruent right triangles are given. Once can cut one of them along an altitude and repeat the operation several times with the newly formed triangles. Prove that there are always two congruent triangles.

Problem 10. Let $n \geqslant 4$. An n-gon is inscribed in a circle, and dissected into $n - 2$ triangles by drawing diagonals. Prove that the sum of the radii of the inscribed circles of the triangles does not depend on the dissection.