GEOMETRY AND TRIGONOMETRY

ROBERT HOUGH

Problem 1. For any 3-dimensional vectors u, v, w, prove the identity

$$u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0.$$

Problem 2. Given two triangles ABC and $A'B'C'$ with the same centroid, prove that one can construct a triangle from the segments AA', BB', CC'.

Problem 3. Let $A_1, A_2, ..., A_n$ be distinct points in the plane, and let m be the number of midpoints of all segments they determine. What is the smallest possible value of m?

Problem 4. Let $A_1A_2...A_n$ be a regular polygon with circumradius equal to 1. Find the maximum value of $\prod_{k=1}^n PA_k$ as P ranges over the circumcircle.

Problem 5. Let $A_0, A_1, ..., A_{n-1}$ be the vertices of a regular n-gon inscribed in the unit circle. Prove that

$$A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = n.$$

Problem 6. Find all regular polygons that can be inscribed in an ellipse with unequal semiaxes.

Problem 7. Prove that the plane

$$\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 1$$

is tangent to the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Problem 8. Through a point M on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

take planes perpendicular to the axes Ox, Oy, Oz. Let the areas of the planar sections be S_x, S_y, S_z. Prove that the sum $aS_x + bS_y + cS_z$ is independent of M.

Problem 9. Let n be a positive integer. Prove that if the vertices of a $(2n+1)$-dimensional cube all have integer coordinates, then the side length of the cube is an integer, too.

Problem 10. Prove the identity

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + ... = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}.$$

Problem 11. Compute the sum

$$\binom{n}{1} \cos x + \binom{n}{2} \cos 2x + \cdots + \binom{n}{n} \cos nx.$$

Problem 12. Prove the identities (angles are in degrees) $\prod_{n=1}^{24} \sec(2^n) = -2^{24} \tan 2$ and $\prod_{n=2}^{25}(2\cos(2^n) - \sec(2^n)) = -1$.