COMBINATORICS

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Problem 1. For each permutation $a_1, a_2, ..., a_{10}$ of $1, 2, ..., 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$

Find the average value of all such sums.

Problem 2. Consider the sequences of real numbers $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_n$, and let σ be a non-trivial permutation of $\{1, 2, ..., n\}$. Prove that

$$\sum_{i=1}^{n} (x_i - y_i)^2 < \sum_{i=1}^{n} (x_i - y_{\sigma(i)})^2.$$

Problem 3. Given some positive real numbers $a_1 < a_2 < \cdots < a_n$ find all permutations σ with the property that

$$a_1 a_{\sigma(1)} < a_2 a_{\sigma(2)} < \cdots < a_n a_{\sigma(n)}.$$

Problem 4. 1981 points lie inside a cube of side length 9. Prove that there are two points within a distance less than 1.

Problem 5. Given a set M of $n \geqslant 3$ points in the plane such that any three points in M can be covered by a disk of radius 1, prove that the entire set M can be covered by a disk of radius 1.

Problem 6. For an arithmetic progression $a_1, a_2, ..., a_n, ...$ let $S_n = a_1 + \cdots + a_n$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} S_{n+1}.$$

Problem 7. Show that for any positive integer n, the numbers

$$S_n = \binom{2n+1}{0} 2^{2n} + \binom{2n+1}{2} 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} 3^n$$

is the sum of two consecutive perfect squares.

Problem 8. Compute the sum $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^m \binom{n}{m}$.

Problem 9. Let p be an odd prime number. Find the number of subsets of $\{1, 2, ..., p\}$ with sum of elements divisible by p.

Problem 10. Find in closed form

$$1 \cdot 2 \binom{n}{2} + 2 \cdot 3 \binom{n}{3} + \cdots + (n-1)n \binom{n}{n}.$$

Problem 11. Prove the combinatorial identity

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

Problem 12. Let E be a set with n elements and F a set with $p \leqslant n$ elements. How many surjective functions are there from E to F?