REAL ANALYSIS

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Problem 1. Does $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{\cos x}}$ exist?

Problem 2. For two positive integers m and n compute

$$\lim_{x\to 0}\frac{(\cos x)^{\frac{1}{m}}-(\cos x)^{\frac{1}{n}}}{x^2}.$$

Problem 3. Let $f(x)=\sum_{k=1}^n a_k\sin kx$, with $a_1,a_2,...,a_n\in\mathbb{R}$, $n\geqslant 1$. Prove that if $f(x)\leqslant|\sin x|$ for all $x\in\mathbb{R}$, then $|\sum_{k=1}^n ka_k|\leqslant 1$.

Problem 4. Does there exist a continuous function $f:[0,1]\to\mathbb{R}$ that assumes every element of its range an even finite number of times?

Problem 5. Let $f:I\to\mathbb{R}$ be a function defined on an interval. Show that if f has the intermediate value property and for any $y\in\mathbb{R}$ the set $f^{-1}(y)$ is closed, then f is continuous.

Problem 6. For any real number $\lambda\geqslant 1$, denote by $f(\lambda)$ the real solution to the equation $x(1+\ln x)=\lambda$. Prove that

$$\lim_{\lambda\to\infty}\frac{f(\lambda)}{\frac{\lambda}{\ln\lambda}}=1.$$

Problem 7. Let $f:[a,b]\to\mathbb{R}$ be a function, continuous on $[a,b]$ and differentiable on (a,b). Let (α,β) be a point on the line passing through the points $(a,f(a))$, $(b,f(b))$ with $\alpha\notin[a,b]$. Prove that there exists a line passing through (α,β) that is tangent to the graph of f.

Problem 8. Let $0<a<b$ and $t_i\geqslant 0$, $i=1,2,...,n$. Prove that for any $x_1,x_2,...,x_n\in[a,b]$,

$$\left(\sum_{i=1}^n t_ix_i\right)\left(\sum_{i=1}^n\frac{t_i}{x_i}\right)\leqslant\frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n t_i\right)^2.$$

Problem 9. Let a_i, $i=1,2,...,n$, be nonnegative numbers with $\sum_{i=1}^n a_i=1$, and let $0<x_i\leqslant 1$, $i=1,2,...,n$. Prove that

$$\sum_{i=1}^n\frac{a_i}{1+x_i}\leqslant\frac{1}{1+x_1^{a_1}\cdots x_n^{a_n}}.$$

Problem 10. Let $0<x_i<\pi$, $i=1,2,...,n$, and set $x=\frac{x_1+\cdots+x_n}{n}$. Prove that

$$\prod_{i=1}^n\left(\frac{\sin x_i}{x_i}\right)\leqslant\left(\frac{\sin x}{x}\right)^n.$$

Problem 11. Let a and b be positive real numbers. Compute the integral

$$\int_a^b\frac{e^{\frac{x}{a}}-e^{\frac{b}{x}}}{x}dx.$$

Problem 12. Compute
$$\int_0^\infty \frac{\ln x}{x^2+a^2}dx,$$
where $a > 0$.

Problem 13. Let $P(x)$ be a polynomial with real coefficients. Prove that
$$\int_0^\infty e^{-x}P(x)dx = P(0) + P'(0) + P''(0) + ...$$

Problem 14. Compute
$$\lim_{n\to\infty}\left(\frac{2^{\frac{1}{n}}}{n+1} + \frac{2^{\frac{2}{n}}}{n+\frac{1}{2}} + \cdots + \frac{2^{n/n}}{n+\frac{1}{n}}\right).$$