ALGEBRA

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Problem 1. Let a, b, c, d be real numbers such that $c \neq 0$ and $ad - bc = 1$. Prove that there exist u and v such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Problem 2. Prove that for any integers $x_1, x_2, ..., x_n$ and any positive integers $k_1, ..., k_n$ the determinant

$$\det \begin{pmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_n^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_n^{k_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k_n} & x_2^{k_n} & \cdots & x_n^{k_n} \end{pmatrix}$$

is divisible by $n!$.

Problem 3. Find the inverse of the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Problem 4. Show that the Hilbert matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

is invertible and the sum of the entries of the inverse is n^2.

Problem 5. Let A and B be 3×3 matrices. Prove that

$$\det(AB - BA) = \frac{\operatorname{tr}((AB - BA)^3)}{3}.$$

Problem 6. Let $(G, +)$ and $(G, *)$ be two group structures defined on the same set G. Assume the two groups have the same identity element, and their group laws satisfy

$$a * b = (a + a) + (a + b)$$

for all $a, b \in G$. Prove that the binary operations are the same, and that the group is Abelian.

Problem 7. Assume a and b are elements of a group with identity element e satisfying $(aba^{-1})^n = e$ for some positive integer n. Prove that $b^n = e$.

Problem 8. Given Γ a finite multiplicative group of matrices with complex entries, denote by M the sum of the matrices in Γ. Prove that $\det M$ is an integer.

Problem 9. Show that infinitely many powers of 2 start with the digit 7 in base 10.

Problem 10. Let R be a nontrivial ring with identity, and $M = \{x \in R : x = x^2\}$ the set of idempotents. Prove that if M is finite it has an even number of elements.

Problem 11. Let R be a ring with identity with the property $(xy)^2 = x^2y^2$ for all $x, y \in R$. Show that R is commutative.