POLYNOMIALS

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Problem 1. Given the polynomial $P(x, y, z)$ prove the polynomial

$$Q(x, y, z) = P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x)$$

is divisible by $(x - y)(y - z)(z - x)$.

Problem 2. Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$ counted without multiplicities.

Problem 3. Let $P(x)$ be a polynomial of degree n. Knowing that $P(k) = \frac{k}{k+1}$, $k = 0, 1, ..., n$ find $P(m)$ for $m > n$.

Problem 4. Solve the system $x + y + z = 1, xyz = 1$ knowing x, y, z complex numbers of absolute value 1.

Problem 5. Prove that for every positive integer n,

$$\tan\frac{\pi}{2n+1}\tan\frac{2\pi}{2n+1}\cdots\tan\frac{n\pi}{2n+1} = \sqrt{2n+1}.$$

Problem 6. Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients of degree greater than or equal to 1, and satisfying $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Prove the polynomials are equal.

Problem 7. Prove that the zeros of the polynomial $P(z) = z^7 + 7z^4 + 4z + 1$ lie inside the disk of radius 2 centered at the origin.

Problem 8. For $a \neq 0$ a real number and $n > 2$ an integer, prove that every nonreal zero z of the polynomial $x^n + ax + 1$ satisfies the inequality $|z| \geqslant \left(\frac{1}{n-1}\right)^{\frac{1}{n}}$.

Problem 9. Let $a \in \mathbb{C}$ and $n \geqslant 2$. Prove that the polynomial equation $ax^n + x + 1 = 0$ has a root of absolute value less than or equal to 2.

Problem 10. Prove that the polynomial

$$P(x) = x^{101} + 101x^{100} + 102$$

is irreducible in $\mathbb{Z}[x]$.

Problem 11. Prove that for every prime number p the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Z}[x]$.