SEQUENCES AND SERIES

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Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x+\sqrt{2}) = f(x+\sqrt{3})$ for all x. Prove that f is constant.

Problem 2. Prove that the sequence $(\sin n)_n$ is dense in $[-1, 1]$.

Problem 3. Define the sequence $(a_n)_{n\geqslant 0}$ by $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 6$, and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \qquad n \geqslant 0.$$

Prove that n divides a_n for all $n \geqslant 1$.

Problem 4. The sequence $a_0, a_1, a_2, ...,$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all nonnegative integers m, n with $m \geqslant n$. If $a_1 = 1$, determine a_n.

Problem 5. Let $p(x) = x^2 - 3x + 2$. Show that for any positive integer n there exist unique numbers a_n and b_n such that the polynomial $q_n(x) = x^n - a_n x - b_n$ is divisible by $p(x)$.

Problem 6. The sequence $(x_n)_n$ is defined by $x_1 = 4, x_2 = 19$, and for $n \geqslant 2$, $x_{n+1} = \left\lceil \frac{x_n^2}{x_{n-1}} \right\rceil$, the smallest integer greater than or equal to $\frac{x_n^2}{x_{n-1}}$. Prove that $x_n - 1$ is always a multiple of 3.

Problem 7. Let $(x_n)_{n\geqslant 1}$ be a sequence of real numbers satisfying

$$x_{n+m} \leqslant x_n + x_m, \qquad n, m \geqslant 1.$$

Show that $\lim_n \frac{x_n}{n}$ exists and is equal to $\inf_{n\geqslant 1} \frac{x_n}{n}$.

Problem 8. Prove that the sequence $(a_n)_{n\geqslant 1}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1), \qquad n \geqslant 1,$$

is convergent.

Problem 9. Let $(a_n)_{n\geqslant 1}$ be a decreasing sequence of positive numbers converging to 0. Prove that the series $S = a_1 - a_2 + a_3 - a_4 + ...$ is convergent.

Problem 10. Let a_0, b_0, c_0 be real numbers. Define the sequences $(a_n)_n$, $(b_n)_n$, $(c_n)_n$ recursively by

$$a_{n+1} = \frac{a_n + b_n}{2}, \; b_{n+1} = \frac{b_n + c_n}{2}, \; c_{n+1} = \frac{c_n + a_n}{2}, \qquad n \geqslant 0.$$

Prove the sequences are convergent and find the limit.

Problem 11. Show that if the series $\sum a_n$ converges, where $(a_n)_n$ is a decreasing sequence, then $\lim_n n a_n = 0$.

Problem 12. Let t and ϵ be real numbers with $|\epsilon| < 1$. Prove that the equation $x - \epsilon \sin x = t$ has a unique real solution.

Problem 13. Prove that for $n \geqslant 2$, the equation $x^n + x - 1 = 0$ has a unique root in the interval $[0, 1]$. If x_n denotes this root, prove that the sequence $(x_n)_n$ is convergent and find its limit.

Problem 14. Let p be a real number, $p \neq 1$. Compute

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}.$$

Problem 15. Consider the polynomial $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, $a_i > 0$, $i = 0, 1, 2, ..., m$. Denote by A_m and G_m the arithmetic and geometric means of $P(1), P(2), ..., P(n)$. Prove that

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e^m}{m+1}.$$

Problem 16. Given a sequence $(a_n)_n$ such that for any $\gamma > 1$ the subsequence $a_{\lfloor \gamma^n \rfloor}$ converges to 0, does it follow that the sequence $(a_n)_n$ itself converges to 0?

Problem 17. Let $(a_n)_{n\geqslant 0}$ be a strictly decreasing sequence of positive numbers, and let z be a complex number of absolute value less than 1. Prove that the sum

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

is not equal to 0.

Problem 18. Let w be an irrational number with $0 < w < 1$. Prove that w has a unique convergent expansion of the form

$$w = \frac{1}{p_0} - \frac{1}{p_0 p_1} + \frac{1}{p_0 p_1 p_2} - \cdots$$

where $p_0, p_1, p_2, ...$ are integers, $1 \leqslant p_0 < p_1 < p_2 < ....$

Problem 19. For a nonnegative integer k, define $S_k(n) = 1^k + 2^k + \cdots + n^k$. Prove that

$$1 + \sum_{k=0}^{r-1} \binom{r}{k} S_k(n) = (n+1)^r.$$

Problem 20. Evaluate in closed form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m!n!}{(m+n+2)!}.$$

Problem 21. Evaluate in closed form

$$\sum_{k=0}^{n} (-1)^k (n-k)!(n+k)!.$$