PROBABILITY

ROBERT HOUGH

Problem 1. Show that for any real numbers $a_1, a_2, ..., a_n$ we have the bound

$$\left|\sum_{j=1}^{n} a_j\right|^2 + \left|\sum_{j=1}^{n}(-1)^j a_j\right|^2 \leqslant (n+2)\sum_{j=1}^{n} a_j^2.$$

Can you give a sharper bound if n is even?

Problem 2. Prove that for complex a_k and b_k, $1 \leqslant k \leqslant n$, one has

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{1\leqslant j<k\leqslant n} |\overline{a}_j b_k - a_k \overline{b}_j|^2.$$

Problem 3. If x, y, z are non-negative real numbers for which $x + y + z = 1$, prove

$$\frac{1}{4} \leqslant x^3 + y^3 + z^3 + 6xyz.$$

Problem 4. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Find the probability that $\sqrt{2+\sqrt{3}} \leqslant |v + w|$.

Problem 5. What is the probability that a permutation of the first n positive integers has the numbers 1 and 2 in the same cycle.

Problem 6. An unbiased coin is tossed n times. Find a formula, in closed form, for the expected value of $|H - T|$, where H is the number of heads, and T is the number of tails.

Problem 7. Prove the identities

$$\sum_{k=1}^{n} \frac{1}{(k-1)!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = 1, \qquad \sum_{k=1}^{n} \frac{k}{(k-1)!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = 2.$$

Problem 8. Given the independent events $A_1, A_2, ..., A_n$ with probabilities $p_1, p_2, ..., p_n$, find the probability that an odd number of these evects occurs.

Problem 9. A coin is tossed n times. What is the probability that two heads will turn up in succession somewhere in the sequence?

Problem 10. We play the coin tossing game in which if tosses match, I get both coins; if they differ, you get both. You have m coins, I have n. What is the expected length of the game?

Problem 11. Let α and β be given positive real numbers, with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

Problem 12. What is the probability that three points selected at random on a circle lie on a semicircle?