POLYNOMIALS

ROBERT HOUGH

Problem 1. Show that if $f(x)$ is a polynomial whose degree is less than n, then the fraction $\frac{f(x)}{(x-x_1)(x-x_2)\cdots(x-x_n)}$ where $x_1, x_2, ..., x_n$ are n distinct numbers, can be represented as a sum of n partial fractions $\frac{A_1}{x-x_1} + \cdots + \frac{A_n}{x-x_n}$ where $A_1, ..., A_n$ are constants.

As an application, let $f(x)$ be a monic polynomial of degree n with distinct zeros $x_1, x_2, ..., x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that $\sum_{j=1}^n \frac{g(x_j)}{f'(x_j)} = 1$.

Problem 2. If $x_1, x_2, ..., x_n$ are distinct numbers, and $y_1, y_2, ..., y_n$ are any numbers, prove that there is a unique polynomial $P(x)$ of degree at most $n-1$ such that $P(x_j) = y_j$, $j = 1, 2, ..., n$.

Problem 3. If $n > 1$, show that $(x+1)^n - x^n - 1 = 0$ has a multiple root if and only if $n-1$ is divisible by 6.

Problem 4. Find all polynomials whose coefficients are equal to either 1 or -1 and whose zeros are all real.

Problem 5. Prove that for every positive integer n,

$$\tan \frac{\pi}{2n+1} \tan \frac{2\pi}{2n+1} \cdots \tan \frac{n\pi}{2n+1} = \sqrt{2n+1}.$$

Problem 6. Determine all polynomials $P(x)$ with real coefficients satisfying $(P(x))^n = P(x^n)$ for all $x \in \mathbb{R}$, where $n > 1$ is a fixed integer.

Problem 7. Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients of degree greater than or equal to 1 with the property that $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Prove that the polynomials are equal.

Problem 8. Let $P(x)$ be a polynomial of degree $n \geqslant 3$ whose zeros $x_1 < x_2 < \cdots < x_n$ are real. Prove that

$$P'\left(\frac{x_1 + x_2}{2}\right) P'\left(\frac{x_{n-1} + x_n}{2}\right) \neq 0.$$

Problem 9. Let $a_1, ..., a_n$ be positive real numbers. Prove that the polynomial $P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n$ has a unique positive zero.

Problem 10. For a polynomial $P(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$, with distinct real zeros $x_1 < x_2 < \cdots < x_n$, $n \geqslant 3$, we set $\delta(P(x)) = \min_i (x_{i+1} - x_i)$. Prove $\delta(P'(x)) > \delta(P(x))$.

Problem 11. Associate to a prime the polynomial whose coefficients are the decimal digits of the prime. Prove that this polynomial is always irreducible over $\mathbb{Z}[x]$.

Problem 12. Let $p < m$ be two positive integers. Prove that

$$\det \begin{pmatrix} \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{p} \\ \binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{p} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+p}{0} & \binom{m+p}{1} & \cdots & \binom{m+p}{p} \end{pmatrix} = 1.$$