PIGEONHOLE AND INVARIANTS

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Problem 1. Let $a_1, a_2, ..., a_n$ represent an arbitrary arrangement of the numbers $1, 2, ..., n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

is even.

Problem 2. Place a knight on a 4-by-n chessboard. Is it possible, in $4n$ consecutive knight moves, to visit each square of the board and return to the original position?

Problem 3. Let X be a real number. Prove that among the numbers $X, 2X, 3X, ..., (n-1)X$ some number differs from an integer by at most $\frac{1}{n}$.

Problem 4. Let p be a prime number and a, b, c integers such that a and b are not divisible by p. Prove that the equation $ax^2 + by^2 = c \bmod p$ has integer solutions.

Problem 5. Show that there is a positive term of the Fibonacci sequence that is divisible by 1000.

Problem 6. Let $P_1, ..., P_{2n}$ be a permutation of the vertices of a regular polygon. Prove that the closed polygonal line $P_1P_2...P_{2n}$ contains a pair of parallel segments.

Problem 7. Consider a planar region of area 1, obtained as the union of finitely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least $\frac{1}{9}$.

Problem 8. Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that

$$n(r) < 2\pi r^{\frac{2}{3}}.$$

Problem 9. In some country all roads between cities are one-way and such that once you leave a city you cannot return to it again. Prove that there exists a city into which all roads enter and a city from which all roads exit.

Problem 10. The positive integers are colored by two colors. Prove that there exists an infinite sequence of positive integers $k_1 < k_2 < ...$ such that the sequence $2k_1 < k_1 + k_2 < 2k_2 < k_2 + k_3 < ...$ all have the same color.

Problem 11. On an arbitrarily large chessboard consider a generalized knight that can jump p squares in one direction and q in the other, $p, q > 0$. Show that such a knight can only return to its original position after an even number of moves.