# NUMBER THEORY

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*Problem* 1. Prove that there is no infinite arithmetic progression whose terms are all perfect squares.

*Problem* 2. Express $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$ in terms of $n$ and $a = \lfloor \sqrt{n} \rfloor$.

*Problem* 3. Prove that for any real number $x$ and for any positive integer $n$,

$$\lfloor nx \rfloor \geqslant \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor nx \rfloor}{n}.$$

*Problem* 4. Let $A$ be the set of positive integers representable in the form $a^2 + 2b^2$ for integers $a, b$ with $b \neq 0$. Show that if $p^2 \in A$ for a prime $p$, then $p \in A$.

*Problem* 5. Prove that among any three distinct integers we can find two, say $a$ and $b$, such that the number $a^3b - ab^3$ is a multiple of 10.

*Problem* 6. The last four digits of a perfect square are equal. Prove that they are all equal to zero.

*Problem* 7. Given a positive integer $n > 1000$, add the residues of $2^n$ modulo each of the numbers $1, 2, 3, ..., n$. Prove that this sum is greater than $2n$.

*Problem* 8. Show that if $n$ has $p - 1$ digits all equal to 1, where $p$ is a prime not equal to $2, 3$ or 5, then $n$ is divisible by $p$.

*Problem* 9. Prove that for any prime $p > 17$, the number

$$p^{32} - 1$$

is divisible by 16320.

*Problem* 10. Let $p$ be an odd prime number. Show that the equation $x^2 \equiv -1 \bmod p$ has a solution if and only if $p \equiv 1 \bmod 4$.

*Problem* 11. Given the non-zero integers $a$ and $d$, show that the sequence

$$a, a + d, a + 2d, ...$$

contains infinitely many terms that have the same prime factors.

*Problem* 12. Let $P(x)$ be a polynomial with integer coefficients. For any positive integer $m$, let $N(m)$ denote the number of solutions to the equation $P(x) \equiv 0 \bmod m$. Show that if $m_1$ and $m_2$ are coprime integers, then $N(m_1m_2) = N(m_1)N(m_2)$.

*Problem* 13. Let $a, b, c, d$ be integers with the property that for any two integers $m$ and $n$ there exist integers $x$ and $y$ satisfying the system

$$ax + by = m, \qquad cx + dy = n.$$

Prove that $ad - bc = \pm 1$.

*Problem* 14. Find a solution to the Diophantine equation

$$x^2 - (m^2 + 1)y^2 = 1,$$

where $m$ is a positive integer.