MULTIVARIABLE CALCULUS, FUNCTIONAL EQUATIONS

ROBERT HOUGH

Problem 1. Let $P(x, y)$ be a harmonic polynomial divisble by $x^2 + y^2$. Prove that $P(x, y)$ is identically equal to zero.

Problem 2. Prove that for $\alpha, \beta, \gamma \in [0, \frac{\pi}{2})$,

$$\tan\alpha + \tan\beta + \tan\gamma \leqslant \frac{2}{\sqrt{3}} \sec\alpha \sec\beta \sec\gamma.$$

Problem 3. Prove that of all quadrilaterals that can be formed from four given sides, the one that is cyclic has the largest area.

Problem 4. Of all triangles circumscribed about a given circle, find the one with the smallest area.

Problem 5. Find the integral of the function

$$f(x, y, z) = \frac{x^4 + 2y^4}{x^4 + 4y^4 + z^4}$$

over the unit ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leqslant 1\}$.

Problem 6. Compute the integral

$$\iint\limits_D \frac{dxdy}{(x^2 + y^2)^2}$$

where D is the domain bounded by the circles

$$x^2 + y^2 - 2x = 0,\ x^2 + y^2 - 4x = 0,\ x^2 + y^2 - 2y = 0,\ x^2 + y^2 - 6y = 0.$$

Problem 7. Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n = m$ be positive integers. Denote by b_k the number of those a_i for which $a_i \geqslant k$. Prove that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_m.$$

Problem 8. Let $|x| < 1$. Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{1}{t} \ln(1 - t) dt.$$

Problem 9. Compute

$$\oint\limits_C y^2 dx + z^2 dy + x^2 dz$$

where C is the Viviani curve, defined as the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with the cylinder $x^2 + y^2 = ax$.

Problem 10. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable functions that are constant along the lines that pass through the origin. Prove that on the unit ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leqslant 1\}$,

$$\iiint\limits_B f \Delta g dV = \iiint\limits_B g \Delta f dV.$$

Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Problem 11. Let

$$G(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0 \right).$$

Prove or disprove that there is a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$,

$$F(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties

(1) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;
(2) $\operatorname{curl} F = 0$ for all $(x,y,z) \neq (0,0,0)$
(3) $F(x,y,0) = G(x,y)$.

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-zero function, satisfying the equation

$$f(x+y) = f(x)f(y), \qquad x, y \in \mathbb{R}.$$

Prove that there exists $c > 0$ such that $f(x) = c^x$ all $x \in \mathbb{R}$.

Problem 13. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + f(y) + f(x)f(y), \qquad x, y \in \mathbb{R}.$$

Problem 14. Find the continuous $\phi, f, g, h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\phi(x+y+z) = f(x) + g(y) + h(z),$$

for all real numbers x, y, z.

Problem 15. Let f and g be differentiable functions on the real line satisfying the equation

$$(f^2+g^2)f' + (fg)g' = 0.$$

Prove that f is bounded.

Problem 16. Let n be a positive integer. Show that the equation

$$(1-x^2)y'' - xy' + n^2 y = 0$$

admits as a particular solution an nth-degree polynomial.