METHODS OF PROOF

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Problem 1. The number 3 can be expressed as the sum of one or more positive integers, taking order into account, in four ways, $3, 2+1, 1+2, 1+1+1$. Prove that a positive integer n can be so expressed in 2^{n-1} ways.

Problem 2. A real-valued function f, defined on the positive rational numbers, satisfies $f(x+y) = f(x)f(y)$ for all positive rational numbers x and y. Prove that $f(x) = [f(1)]^x$ for all positive rational x.

Problem 3. Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number.

Problem 4. Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.

Problem 5. Every point of three-dimensional space is colored red, green or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

Problem 6. Show that there does not exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(2) = 3$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

Problem 7. Show that the interval $[0, 1]$ cannot be partitioned into two disjoint sets A and B such that $B = A + a$ for some real number a.

Problem 8. Prove that $|\sin nx| \leqslant n|\sin x|$ for any real number x and positive integer n.

Problem 9. Prove that for any real numbers $x_1, ..., x_n$, $|\sin x_1| + \cdots + |\sin x_n| + |\cos(x_1 + x_2 + \cdots + x_n)| \geqslant 1$.

Problem 10. Prove that for any positive integer $n \geqslant 2$ there is a positive integer m that can be written simultaneously as the sum of 2,3 , ..., n non-zero squares of integers.

Problem 11. Prove that any polygon can be dissected into triangles by interior diagonals.

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f\left(\frac{x_1+x_2}{2}\right) = \frac{f(x_1)+f(x_2)}{2}$. Prove that

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \frac{f(x_1) + \cdots + f(x_n)}{n}$$

for any $x_1, ..., x_n$.

Problem 13. Show that if $a_1, ..., a_n$ are nonnegative numbers, then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqslant (1 + (a_1 a_2 \cdots a_n)^{\frac{1}{n}})^n.$$