GEOMETRY AND TRIGONOMETRY

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Problem 1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Assume that f'' exists on $(0,1)$ and $f''(x) + 2f'(x) + f(x) \geqslant 0$ for all $x \in (0,1)$. Prove that $f(x) \leqslant 0$ for all $x \in [0,1]$.

Problem 2. Let M be a point in the plane of triangle ABC. Prove that the centroids of the triangles MAB, MAC, MCB for a triangle similar to ABC.

Problem 3. Find the locus of points P in the interior of a triangle ABC such that the distances from P to the lines AB, BC, and CA are the side lengths of some triangle.

Problem 4. Let $ABCDEF$ be a hexagon inscribed in a circle of radius r. Show that if $AB = CD = EF = r$, then the midpoints of BC, DE, and FA are the vertices of an equilateral triangle.

Problem 5. Prove that in a triangle the orthocenter H, centroid G, and circumcenter O are collinear. Moreover, G lies between H and O and $\frac{OG}{GH} = \frac{1}{2}$.

Problem 6. A cube is rotated about the main diagonal. What kind of surfaces do the edges describe?

Problem 7. Let n be a positive integer. Prove that if the vertices of a $(2n+1)$-dimensional cube have integer coordinates, then the length of the edge of the cube is an integer.

Problem 8. A polyhedron is circumscribed about a sphere. We call a face *big* if the projection of the sphere onto the plane of the face lies entirely within the face. Show that there are at most six big faces.

Problem 9. Let $f : [0,a] \to \mathbb{R}$ be a continuous and increasing function such that $f(0) = 0$. Define by R the region bounded by $f(x)$ and the lines $x = a$ and $y = 0$. Now consider the solid of revolution obtained when R is rotated around the y-axis as a sort of dish. Determine f such that the volume of water the dish can hold is equal to the volume of the dish itself, this happening for all a.

Problem 10. Someone has drawn two squares of side 0.9 inside a disk of radius 1. Prove that the squares overlap.

Problem 11. Show that if all angles of an octagon are equal and all its sides have rational length, then the octagon has a center of symmetry.

Problem 12. Show that if the angles a and b satisfy
$$\tan^2 a \tan^2 b = 1 + \tan^2 a + \tan^2 b$$
then
$$\sin a \sin b = \pm\frac{\sqrt{2}}{2}.$$

Problem 13. Prove the identity
$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2}\cos\frac{n\pi}{4}, \qquad n \geqslant 1.$$

Problem 14. Compute the sum

$$\binom{n}{1}\cos x + \binom{n}{2}\cos 2x + \cdots + \binom{n}{n}\cos nx.$$

Problem 15. Find

$$\cos(\alpha)\cos(2\alpha)\cdots\cos(999\alpha)$$

with $\alpha = \frac{2\pi}{1999}$.

Problem 16. Prove that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin 1^\circ}.$$

Problem 17. Obtain explicit values for the following series:

(1) $\sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$
(2) $\sum_{n=1}^{\infty} \arctan \frac{8n}{n^4 - 2n^2 + 5}$.

Problem 18. In a circle of radius 1 a square is inscribed. A circle is inscribed in the suqre, and then a regular octagon inside the circle. The procedure continues, doubling each time the number of sides of the polygon. Find the limit of the lengths of the radii of the circles.

Problem 19. Prove that

$$\left(1 - \frac{\cos 61^\circ}{\cos 1^\circ}\right)\left(1 - \frac{\cos 62^\circ}{\cos 2^\circ}\right)\cdots\left(1 - \frac{\cos 119^\circ}{\cos 59^\circ}\right) = 1.$$