CONTINUITY, DERIVATIVES, INTEGRALS

ROBERT HOUGH

Problem 1. Does there exist a continuous function $f : [0,1] \to \mathbb{R}$ that assumes every element of its range an even (finite) number of times?

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Prove that the system

$$x = f(y), \; y = f(z), \; z = f(x)$$

has a unique solution.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geqslant |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the range of f is all of $\mathbb{R}$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For $x \in \mathbb{R}$ we define

$$g(x) = f(x) \int_0^x f(t)dt.$$

Show that if g is a nonincreasing function, then f is identically equal to zero.

Problem 5. Let f be a function having a continuous derivative on $[0,1]$ and with the property that $0 < f'(x) \leqslant 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left[\int_0^1 f(x)dx\right]^2 \geqslant \int_0^1 [f(x)]^3 dx.$$

Give an example with equality.

Problem 6. Let α be a real number such that n^α is an integer for every positive integer n. Prove that α is a nonnegative integer.

Problem 7. Show that if a function $f : [a,b] \to \mathbb{R}$ is convex, then it is continuous on (a,b).

Problem 8. Let $0 < a < b$ and t_i, $i = 1, 2, ..., n$. Prove that for any $x_1, x_2, ..., x_n \in [a,b]$,

$$\left(\sum_{i=1}^n t_i x_i\right)\left(\sum_{i=1}^n \frac{t_i}{x_i}\right) \leqslant \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n t_i\right)^2.$$

Problem 9. Prove that for any natural number $n \geqslant 2$ and any $|x| \leqslant 1$

$$(1+x)^n + (1-x)^n \leqslant 2^n.$$

Problem 10. Let a_i, $i = 1, 2, ..., n$, be nonnegative numbers with $\sum_{i=1}^n a_i = 1$, and let $0 < x_i \leqslant 1$, $i = 1, 2, ..., n$. Prove that

$$\sum_{i=1}^n \frac{a_i}{1+x_i} \leqslant \frac{1}{1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

Problem 11. Compute the integral

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)dx.$$

Problem 12. Let $P(x)$ be a polynomial with real coefficients. Prove that

$$\int_0^\infty e^{-x}P(x)dx = P(0) + P'(0) + P''(0) + \ldots.$$

Problem 13. Determine the continuous functions $f : [0,1] \to \mathbb{R}$ that satisfy

$$\int_0^1 f(x)(x - f(x))dx = \frac{1}{12}.$$

Problem 14. Let f be a non-increasing function on the interval $[0,1]$. Prove that for any $\alpha \in (0,1)$,

$$\alpha \int_0^1 f(x)dx \leqslant \int_0^\alpha f(x)dx.$$

Problem 15. Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|dxdy \geqslant \int_0^1 |f(x)|dx.$$

Problem 16. Prove that for $|x| < 1$,

$$\arcsin x = \sum_{k=0}^\infty \frac{1}{2^{2k}(2k+1)} \binom{2k}{k} x^{2k+1}.$$

Problem 17. Prove that for every $0 < x < 2\pi$ the following formula is valid:

$$\frac{\pi - x}{2} = \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots.$$

Derive the formula

$$\frac{\pi}{4} = \sum_{k=1}^\infty \frac{\sin(2k-1)x}{2k-1}, \qquad x \in (0,\pi).$$

Problem 18. Use the Fourier series of the function of period 1 defined by $f(x) = \frac{1}{2} - x$ for $0 \leqslant x < 1$ to prove Euler's formula

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$