COMBINATORICS

ROBERT HOUGH

Problem 1. Let $A_1, A_2, A_3, ...$ be finite sets and $f_i : A_{i+1} \to A_i$ a sequence of maps. Prove that there is a sequence $x_1, x_2, x_3, ...$ with $x_i \in A_i$ and $f_i(x_{i+1}) = x_i$ for all i.

Problem 2. Consider the sequences of real numbers $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_n$, and let σ be a nontrivial permutation of the set $\{1, 2, ..., n\}$. Prove that

$$\sum_{i=1}^{n}(x_i - y_i)^2 < \sum_{i=1}^{n}(x_i - y_{\sigma(i)})^2.$$

Problem 3. Determine the number of permutations $a_1, a_2, ..., a_{2004}$ of the numbers $1, 2, ..., 2004$ for which

$$|a_1 - 1| = |a_2 - 2| = \cdots = |a_{2004} - 2004| > 0.$$

Problem 4. Given $n > 4$ points in the plane such that no three are collinear, prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

Problem 5. An equilateral triangle of side length n is drawn with sides along a triangular grid of side length 1. What is the maximum number of grid segments on or inside the triangle that can be marked so that no three marked segments form a triangle?

Problem 6. Several chords are constructed in a circle of radius 1. Prove that if every diameter intersects at most k chords, then the sum of the lengths of the chords is less than $k\pi$.

Problem 7. Denote by V the number of vertices of a convex polyhedron, and by Σ the sum of the (planar) angles of its faces. Prove that $2\pi V - \Sigma = 4\pi$.

Problem 8. Let n be a positive integer satisfying the following property: if n dominoes are placed on a 6×6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n.

Problem 9. Let $(F_n)_n$ be the Fibonacci sequence, $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. Prove that for any positive integer n,

$$F_1\binom{n}{1} + F_2\binom{n}{2} + \cdots + F_n\binom{n}{n} = F_{2n}.$$

Problem 10. For a positive integer n, denote by $S(n)$ the number of choices of the sign "+" or "−" such that $\pm 1 \pm 2 \pm \cdots \pm n = 0$. Prove

$$S(n) = \frac{2^{n-1}}{\pi}\int_0^{2\pi} \cos t \cos 2t \cdots \cos nt dt.$$

Problem 11. Let $A_1, A_2, ..., A_n, ...$ and $B_1, B_2, ..., B_n, ...$ be sequences of sets defined by $A_1 = \varnothing$, $B_1 = \{0\}$, $A_{n+1} = \{x + 1 : x \in B_n\}$, $B_{n+1} = (A_n \cup B_n) \backslash (A_n \cap B_n)$. Determine all positive integers n for which $B_n = \{0\}$.

Problem 12. Find in closed form

$$1 \cdot 2\binom{n}{2} + 2 \cdot 3\binom{n}{3} + \cdots + (n-1) \cdot n\binom{n}{n}.$$

Problem 13. Prove the identity

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} 2^k.$$

Problem 14. Prove that the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_m = n$$

is equal to $\binom{m+n-1}{m-1}$.

Problem 15. Let A be a 101-element subset of the set $S = \{1, 2, ..., 1000000\}$. Prove that there exist numbers $t_1, t_2, ..., t_{100}$ in S such that the sets

$$A_j\{x + t_j : x \in A\}, \qquad j = 1, 2, ..., 100,$$

are pairwise disjoint.

Problem 16. Let E be a set with n elements and F a set with p elements, $p \leqslant n$. How many surjective (i.e., onto) functions $f : E \to F$ are there?

Problem 17. Given a graph with n vertices, prove that either it contains a triangle, or there exists a vertex that is the endpoint of at most $\lfloor \frac{n}{2} \rfloor$ edges.