ALGEBRAIC IDENTITIES AND INEQUALITIES

ROBERT HOUGH

Problem 1. Prove that any polynomial with real coefficients that takes only nonnegative values can be written as the sum of the squares of two polynomials.

Problem 2. Factor $5^{1985} - 1$ into a product of three integers, each of which is $> 5^{100}$.

Problem 3. Let $a_1, ..., a_n$ be real numbers such that $a_1 + \cdots + a_n \geqslant n^2$ and $a_1^2 + \cdots + a_n^2 \leqslant n^3 + 1$. Prove that $n - 1 \leqslant a_k \leqslant n + 1$ for all k.

Problem 4. If a, b, c are positive numbers, prove that $9a^2b^2c^2 \leqslant (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)$.

Problem 5. If $a_1 + a_2 + \cdots + a_n = n$ prove that $a_1^4 + a_2^4 + \cdots + a_n^4 \geqslant n$.

Problem 6. Let $f_1, f_2, ..., f_n$ be positive real numbers. Prove that for any real numbers $x_1, ..., x_n$, the quantity

$$f_1x_1^2 + \cdots + f_nx_n^2 \geqslant \frac{(f_1x_1 + \cdots + f_nx_n)^2}{f_1 + \cdots + f_n}.$$

Problem 7. Prove that the finite sequence $a_0, a_1, ..., a_n$ of positive real numbers is a geometric progression if and only if

$$(a_0a_1 + \cdots + a_{n-1}a_n)^2 = (a_0^2 + \cdots + a_{n-1}^2)(a_1^2 + \cdots + a_n^2).$$

Problem 8. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be nonnegative numbers. Show that

$$(a_1...a_n)^{1/n} + (b_1...b_n)^{1/n} \leqslant ((a_1 + b_1)(a_2 + b_2)...(a_n + b_n))^{1/n}.$$

Problem 9. Let $a_1, ..., a_n$ be positive real numbers such that $a_1 + \cdots + a_n < 1$. Prove that

$$\frac{a_1a_2 \cdots a_n(1 - (a_1 + \cdots a_n))}{(a_1 + \cdots + a_n)(1 - a_1) \cdots (1 - a_n)} \leqslant \frac{1}{n^{n+1}}.$$

Problem 10. Consider the positive real numbers $x_1, ..., x_n$ with $x_1x_2 \cdots x_n = 1$. Prove that

$$\frac{1}{n - 1 + x_1} + \cdots + \frac{1}{n - 1 + x_n} \leqslant 1.$$

Problem 11. Let $x_1, x_2, ..., x_n$, $n \geqslant 2$ be positive numbers such that $x_1 + \cdots + x_n = 1$. Prove that

$$\left(1 + \frac{1}{x_1}\right) \cdots \left(1 + \frac{1}{x_n}\right) \geqslant (n + 1)^n.$$

Problem 12. Let $x_1, ..., x_n$ be n real numbers such that $0 < x_j \leqslant \frac{1}{2}$. Prove

$$\frac{\prod_{j=1}^n x_j}{\left(\sum_j x_j\right)^n} \leqslant \frac{\prod_{j=1}^n (1 - x_j)}{\left(\sum_j (1 - x_j)\right)^n}.$$

Problem 13. Let $a_1, ..., a_n, b_1, ..., b_n$ be real numbers such that

$$(a_1^2 + \cdots + a_n^2 - 1)(b_1^2 + \cdots + b_n^2 - 1) > (a_1b_1 + \cdots a_nb_n - 1)^2.$$

Prove that $a_1^2 + \cdots + a_n^2 > 1$ and $b_1^2 + \cdots + b_n^2 > 1$.

Problem 14. Let a, b, c be real numbers. Show that $a \geqslant 0$, $b \geqslant 0$ and $c \geqslant 0$ if and only if $a + b + c \geqslant 0$, $ab + ac + bc \geqslant 0$, and $abc \geqslant 0$.